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Modelling Mortality: the Effect of Small Population Size on Parameter Estimation

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Background and Motivation

Background

- Many mortality models, e.g. Lee-Carter model (1992) and CBD (2006), have been introduced to provide a good fitness for large sized populations
- Actuaries are interested in modelling relatively much smaller populations, e.g.Pension scheme
- Historical data has shown that smaller population has significant variability on its pattern compared with larger population
- Booth (2006) showed poorness of fit of Lee-Carter model to smaller populations

Motivation

- Investigate to what extent information contained in larger population can be adopted

Death count $D(t, x)$ and corresponding exposure $E(t, x)$ of

- Males and Females in England and Wales (EW)
- Males only in Scotland
- Year range: 1961-2011
- Age range: 50-89, last birthday

Note: The limited size of dataset inevitably results in extra uncertainty to the parameter estimates

- Mortality model, M7:

$$\text{logit}q(t,x) = \kappa_t^{(1)} + \kappa_t^{(2)}(x - \bar{x}) + \kappa_t^{(3)}((x - \bar{x}) - \hat{\sigma}_x^2) + \gamma_{t-x}^{(4)}$$

- $\kappa_t^{(i)}$, period effect; $\gamma_{t-x}^{(4)}$, cohort effect
- Let $\theta = (\kappa_t^{(1)}, \kappa_t^{(2)}, \kappa_t^{(3)}, \gamma_{t-x}^{(4)})$
- $$q(t,x) = 1 - \exp[-m(\theta, t, x)]$$
- Assume
$$D(t,x) \sim Po(m(\theta; t, x)E(t,x))$$
- Sampling variation in $D(t,x)$ leads to noise in $\kappa_t^{(i)}$ and $\gamma_c^{(4)}$ estimates

- $E_0(t,x)$, $D_0(t,x)$: the benchmark exposure and corresponding deaths, respectively.
- $\hat{\theta}_0$: the benchmark parameter estimates for $D_0(t,x)$.
- Simulation model

$$D_w(t,x)|\hat{\theta}_0 \sim Po(m(\hat{\theta}_0,t,x)wE_0(t,x))$$

  where $w = 1, 0.1, 0.01$
- Simulate $D_w(t,x) \rightarrow \hat{\theta}_w$
- $D_w(t,x)^j$, $\hat{\theta}^j_w$, where $j = 1, ..., K$
- Let the exposure of England and Wales, $E_{EW}(t,x)$ be the bench mark, i.e. $E_0(t,x) = E_{EW}(t,x)$

- Recall for random variables $X_1, ...X_n$ with pdf or pmf $f(x|\theta)$, the asymptotic distribution of MLE $\hat{\theta}$ is

$$\hat{\theta} \asymp N(\theta_0, \frac{1}{I(\theta_0)})$$

where I is the Fisher information

- In our case,

$$\hat{\theta}_w \asymp N(\hat{\theta}^{EW}, \frac{1}{I(\hat{\theta}^{EW})})$$

where

$$I(\hat{\theta}^{EW}) = w \sum_{t,x} \frac{E^{EW}_{t,x}}{m(t,x,\hat{\theta}^{EW})} (\frac{\partial m(t,x,\theta_w)}{\partial \theta_w})^2$$

- Thus the variation of $\hat{\theta}_w$ is inversely proportional to the population size

(E.g. Kendall, 1991, Volume 2)

- Define likelihood function: $L(x|\theta) = \prod_{i=1}^{n} f(x_i|\theta)$, where $\theta = (\theta_r, \theta_s)$, $r + s = k$.
- Null hypothesis and alternative:

$$H_0 : \theta_r = \theta_{r0};\ \ H_1 : \theta_r \neq \theta_{r0}.$$

- Unconditional maximum of $L(x|\theta)$,

$$L(x \mid \hat{\theta}_r, \hat{\theta}_s)$$

  where $(\hat{\theta}_r, \hat{\theta}_s)$ is MLEs of (θ_r, θ_s)
- Conditional maximum

$$L(x \mid \theta_{r0}, \tilde{\theta}_s(\theta_{r0})).$$

  where $\tilde{\theta}_s(\theta_{r0})$ is the MLEs of θ_s, given H_0 holds
- Log-scaled test statistic

$$\Gamma = -2\log \frac{L(x \mid \theta_{r0}, \tilde{\theta}_s)}{L(x \mid \hat{\theta}_r, \hat{\theta}_s)}$$

Set $s = 0$

- The null hypothesis and alternative:

$$H_0 : \theta_r^w = \theta_{r0}; \;\; H_1 : \theta_r^w \neq \theta_{r0}.$$

  where θ_r^w is the parameters for population w.

- Log-scaled test statistic is

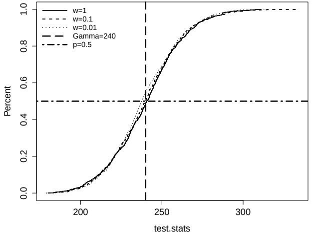
$$\Gamma^w = -2(l(x_w \mid \theta_{r0}) - l(x_w \mid \hat{\theta}_r))$$

- Under H_0, Γ^w has asymptotic χ_r^2 distribution with r degree of freedom.

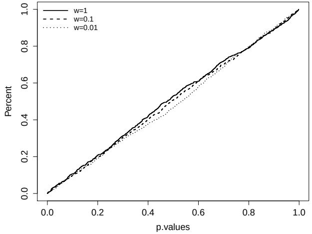
Let

$$\theta_{r0} = \hat{\theta}^{EW};\ \hat{\theta}_r = \hat{\theta}_w$$

(a) Test Statistics, Γ^w



(b) P-values, P_w



Power of hypothesis test:

$$\text{Prob (Reject } H_0 \mid H_1 \text{is True)} = 1 - \text{type II error rate}$$

Define $\hat{\theta}^{(i)}$:

- $\hat{\theta}^{(1)} = (\hat{\kappa}_t^{EW,(1)} + \lambda, \hat{\kappa}_t^{EW,(2)}, \hat{\kappa}_t^{EW,(3)}, \hat{\gamma}_c^{EW,(4)})$
- $\hat{\theta}^{(2)} = (\hat{\kappa}_t^{EW,(1)}, \hat{\kappa}_t^{EW,(2)} + \lambda, \hat{\kappa}_t^{EW,(3)}, \hat{\gamma}_c^{EW,(4)})$
- $\hat{\theta}^{(3)} = (\hat{\kappa}_t^{EW,(1)}, \hat{\kappa}_t^{EW,(2)}, \hat{\kappa}_t^{EW,(3)} + \lambda, \hat{\gamma}_c^{EW,(4)})$
- $\hat{\theta}^{(4)} = (\hat{\kappa}_t^{EW,(1)}, \hat{\kappa}_t^{EW,(2)}, \hat{\kappa}_t^{EW,(3)}, \hat{\gamma}_c^{EW,(4)} + \lambda)$

Testing Systematic Parameter Difference

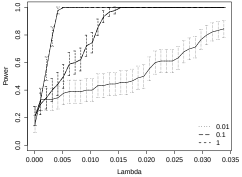
Power of the LR test, Ctd

- Simulate

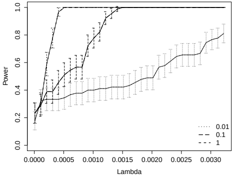
$$D_{t,x}^{w,(i)} \mid \hat{\theta}^{(i)} \sim Po(m(\hat{\theta}^{(i)}, t, x) w E_{t,x}^{EW}), i = 1, 2, 3, 4$$

- $\hat{\theta}_{r,w}^{(i)}$, the MLEs of $D_{t,x}^{w,(i)} \mid \hat{\theta}^{(i)}$
- Propose LR test by setting $\theta_{r0} = \hat{\theta}^{(i)}$; $\hat{\theta}_r = \hat{\theta}_{r,w}^{(i)}$

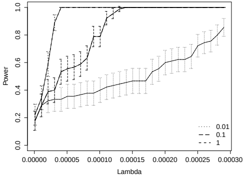
(a) The Power of LR test when $\kappa^{(1)}$ shifted



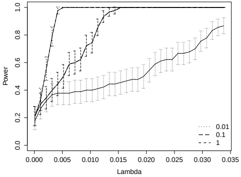
(b) The Power of LR test when $\kappa^{(2)}$ shifted



(c) The Power of LR test when $\kappa^{(3)}$ shifted



(d) The Power of LR test when $\gamma^{(2)}$ shifted



Question: does population A share the same cohort effect, given $\gamma_B^{(4)}$ already estimated

- $\theta_r = \gamma_A^{(4)}$; $\theta_s = (\kappa_t^{(1)}, \kappa_t^{(2)}, \kappa_t^{(3)})$
- Test statistic

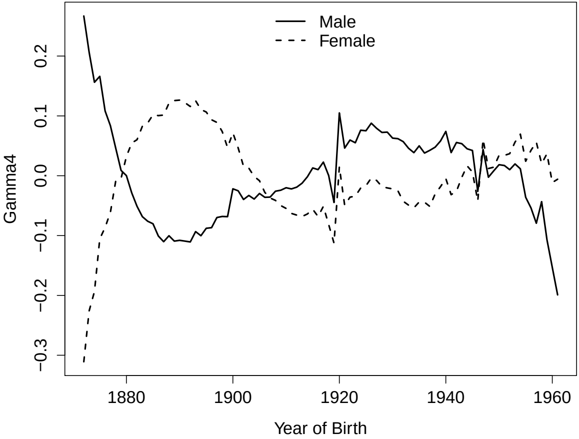
$$\Gamma = -2\log \frac{L(x \mid \theta_{r0}, \tilde{\theta}_s)}{L(x \mid \hat{\theta}_r, \hat{\theta}_s)}$$

Let

- B: EW males
- A1: EW females, $\Gamma_{A1} = 6311 \rightarrow p \simeq 0$
- A2: Scotland males, $\Gamma_{A2} = 193 \rightarrow p \simeq 0$
- However, we see great improvement on the test statistic from A1 to A2.

LR Test for Common Cohort Effect in Two Populations

Testing Common Cohort Effect between Males and Females in England & Wales



Let A, B be males in Scotland and England & Wales respectively

- Both populations have similar pattern.
- Strong linear relationship between cohort effect estimates of Scotland and England & Wales.

Conclusion

- The variation of parameter estimates can be significantly and proportionally increased by reducing the population size.
- For the given dataset, χ^2 provides a good asymptotic approximation for the LR test statistic under null hypothesis.
- The power of LR test is approximately proportional to the population size. We can significantly improve the power by increasing the population size.
- Males and Females in England & Wales; males in England & Wales and Scotland do not have the same cohort effect.