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Internal Model Calibration Using Overlapping Data

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Agenda

- Problem Statement
- Overview
- Cumulant Estimation
- Overlapping vs. Non-Overlapping approach
- Solutions and Alternatives
- Simulation study
- Summary

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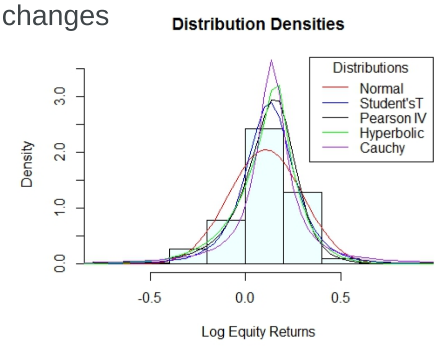
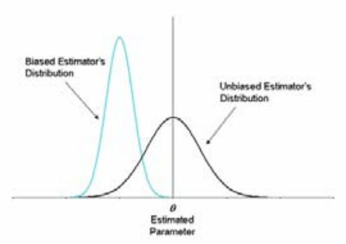
Problem Statement

- Solvency II requires firms to estimate risk distributions under Internal Model
- Limited data available in most cases
- General market practice is to use annual overlapping changes
- Key questions faced by calibrators are
  - Relevance of available data vs. length of data
  - Use overlapping data or non-overlapping data
  - If any adjustment is required for using overlapping data
  - Any alternative to using overlapping data
  - Implications for validation tests



Overview

- Use of Cumulants and Distribution fitting
- Pros and cons of using overlapping and non-overlapping
- Possible solutions to using overlapping data
  - Use of bias corrections such as Nelken or Cochrane Adjustment
  - Use of data annualisation or statistical technique such as "Temporal aggregation"
- Key Conclusions
  - Simulation study of standard processes shows
    - Use of both overlapping and non-overlapping data lead to bias in the data
    - Methods of bias correction lead to increased variance particularly when data is less than 20-30 years
  - Alternative approaches such "temporal aggregation" or "annualisation" are credible but have their own limitations



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Cumulants – What are they?

- Cumulants are properties of random variables and provide alternatives to moments
- First two cumulants are the same as central moments, e.g. mean and variance.
- The 3rd cumulant is also the 3rd central moment.

Cumulants	Central Moments
$\kappa_1 = \mu = \mathbb{E}(X)$	$\mathbb{E}(X)$
$\kappa_2 = \mathbb{E}(X-\mu)^2$	$\mathbb{E}(X-\mu)^2$
$\kappa_3 = \mathbb{E}(X-\mu)^3$	$\mathbb{E}(X-\mu)^3$
$\kappa_4 = \mathbb{E}(X-\mu)^4 - 3\kappa_2^2$	$\mathbb{E}(X-\mu)^4$

- But 4th and higher order cumulants are not equal to central moments.
- Cumulants have some nice properties e.g. for independent variables they are additive.

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Cumulants – Distribution fitting

- Empirical cumulants could be used for distribution fitting.
- **Objective**: Identify distribution whose first four cumulants match the cumulants estimated from data sample (similar to Method of Moments).

or Alternatively

- **Objective**: Identify distribution with the maximum value of log-likelihood function (Maximum Likelihood Estimation).

- Under SII, the emphasis is on the entire risk distribution rather than on 1-in-200 point and therefore
  - Skewness and Kurtosis are equally as important as mean and variance

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Overlapping vs. Non-overlapping

- Cumulants to be estimated from historical data – could be done via use of overlapping annual changes or non-overlapping annual changes

Overlapping Data	Non-overlapping Data
Advantages	
More data points available for calibration	Directly looks at 1-Yr time window
Independent of time-period window selection	Considered theoretically most accurate
Disadvantages	
Data is auto-correlated and not independent	Mostly available data insufficient
Estimates are biased and standard goodness-of-fit tests invalid	Estimates are biased
	Uses less information and hence careful time-window selection needed

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Solutions & Alternatives – Bias corrections

- Sample variance from either non-overlapping or overlapping data is a biased estimator of population variance
  - Bessel's $n-1$ correction for non-overlapping iid data
  - Cochrane (1987) and Nelken & Sun et al (2009) give corrections for overlapping data where the underlying model is iid (eg Brownian motion).
  - No approach will remove bias for arbitrary dependence (mean reversion etc.)
- Allowing overlaps means that more data is used
  - As a consequence, the variance of the overlapping estimators tends to be lower, i.e. the calibration is likely to be more accurate
  - This (reduced estimator variance) doesn't hold for first cumulant estimates (Müller 1993)

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Simulation Study – Overlapping vs Nonoverlapping

- Simulation study assesses the impact on cumulant estimation using annual overlapping and non-overlapping data
- Using a method outlined in Jarvis et al. (“Ersatz Model Tests”), monthly returns are simulated from a known reference distribution.
- The first four cumulants are estimated using both annual overlapping data and non overlapping data
- The bias and the mean square error from these two approaches can be compared with the known exact answers to compare the approaches
- Four different reference models were considered:
  - Brownian motion
  - Negative Inverse Gaussian
  - ARIMA
  - GARCH

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Simulation Study – Background

- For each different reference model:
  - N years of monthly data are simulated;
  - annual overlapping and non overlapping returns calculated
  - the first four cumulants estimated
- This is repeated 1000 times for values of N years from 2 to 50
- The bias and the mean square error from each of the overlapping and non-overlapping approaches are calculated for each value of N
- The bias and mean square error are then compared in the plots on the following slides

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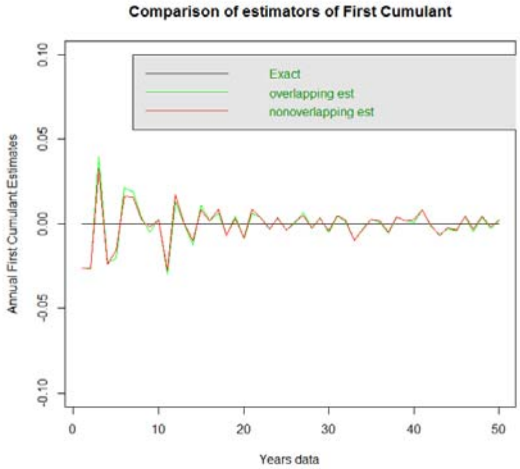
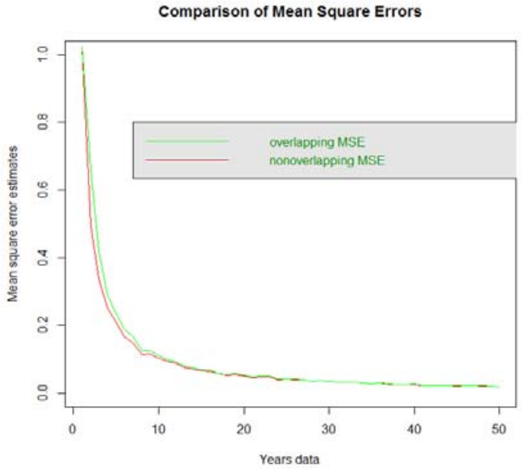
Simulation Study Results – Brownian 1st cumulants





- 1st cumulant:
  - Both the approaches are unbiased and have similar MSE

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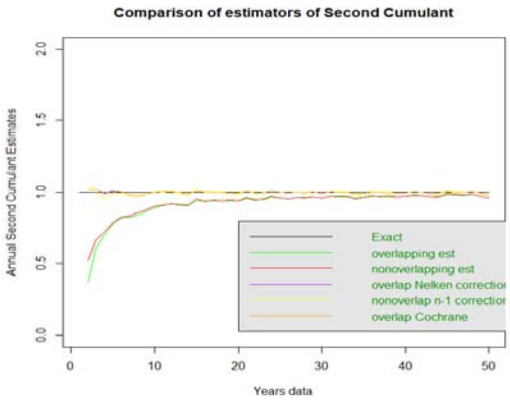
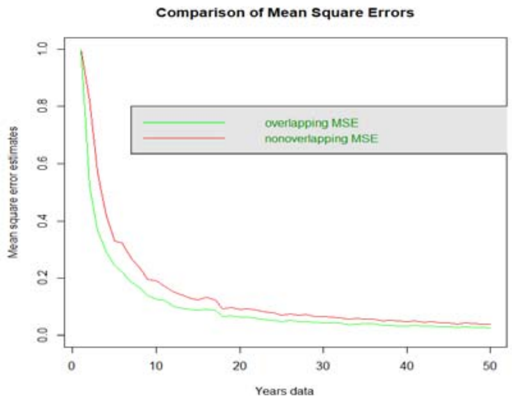
Simulation Study Results – Brownian 2nd cumulants





- 2nd cumulant:
  - Both approaches have similar level of bias, particularly when available data is limited (i.e. less than 20 years)
  - The bias corrections applied to data appear to have removed the bias for both the approaches
  - The MSE appears to be is lower using overlapping data.

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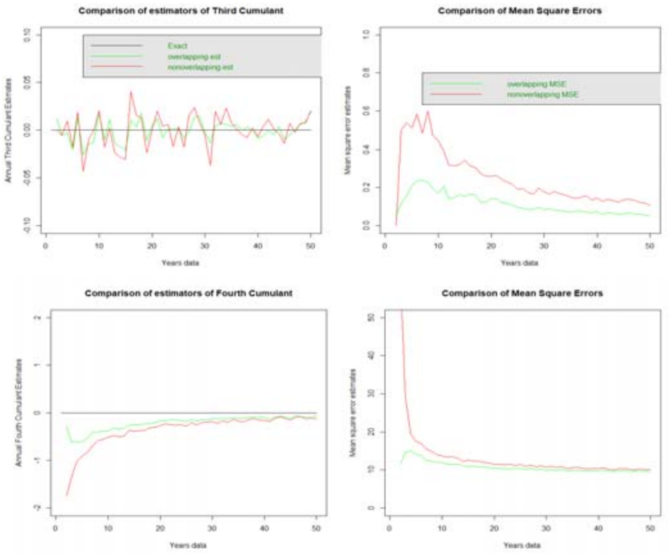
Simulation Study Results Brownian 3rd / 4th Cumulants



- 3rd cumulant:
  - Neither approach appears to have any systemic bias
- 4th cumulant:
  - Both approaches are biased but overlapping has lower downward bias comparatively
  - Overlapping data has lower MSE.
- **Note: Our analysis shows that Normal inverse Gaussian (special case of Levy process) also leads to similar conclusions.**

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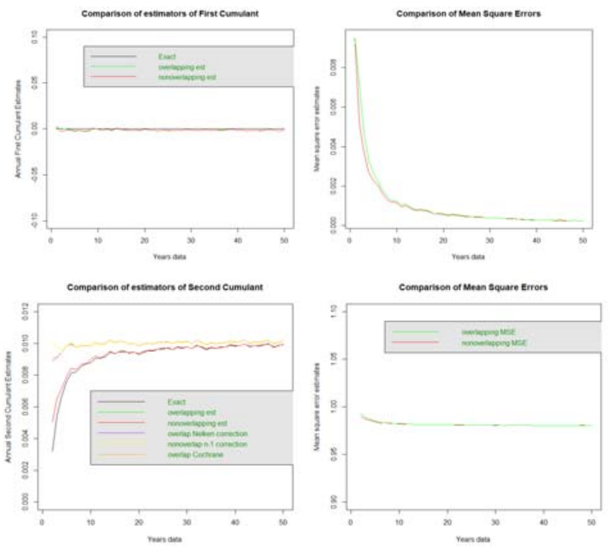
Simulation Study Results – ARIMA 1st & 2nd cumulants



- 1st cumulant:
  - Both the approaches are unbiased and have similar MSE
- 2nd cumulant:
  - Both approaches have similar level of bias, particularly when available data is limited
  - The bias corrections applied to data appear not to have removed the bias completely for small sample sets.
  - The MSE appears to be similar under both the approaches.

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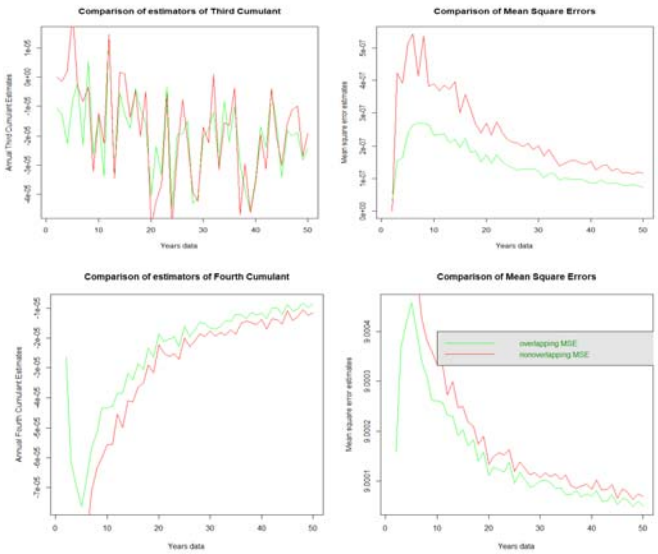
Simulation Study Results – ARIMA 3rd / 4th cumulants



- 3rd cumulant:
  - Neither approach appears to have any materially different bias
  - Non-overlapping data has higher MSE compared to overlapping data
- 4th cumulant:
  - Non-overlapping data has lower bias compared to overlapping data
  - Overlapping data has lower MSE.

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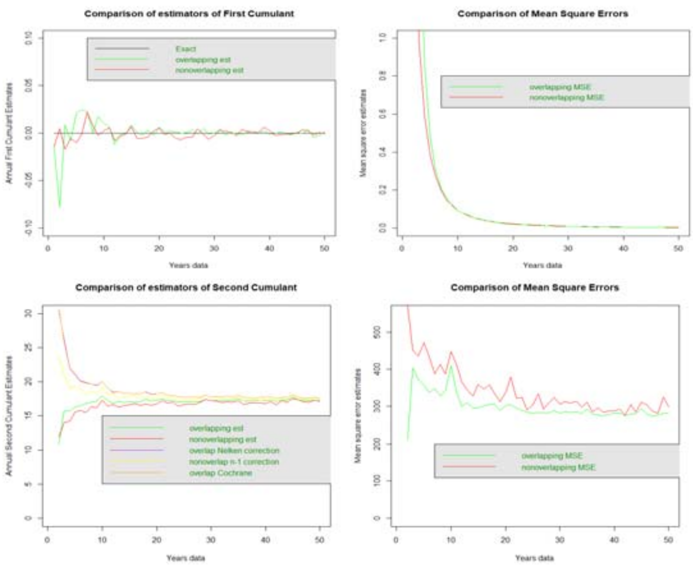
Simulation Study Results – GARCH 1st & 2nd cumulants



- 1st cumulant:
  - Both the approaches have similar level of bias and converges to exact beyond 30 years of data
  - Overlapping data has marginally higher MSE but converges quickly
- 2nd cumulant:
  - Both approaches have similar level of bias, particularly when available data is limited
  - The bias corrections applied to data appear not to have removed the bias completely for small sample sets.
  - The MSE for overlapping data appears to be materially lower than non-overlapping data.

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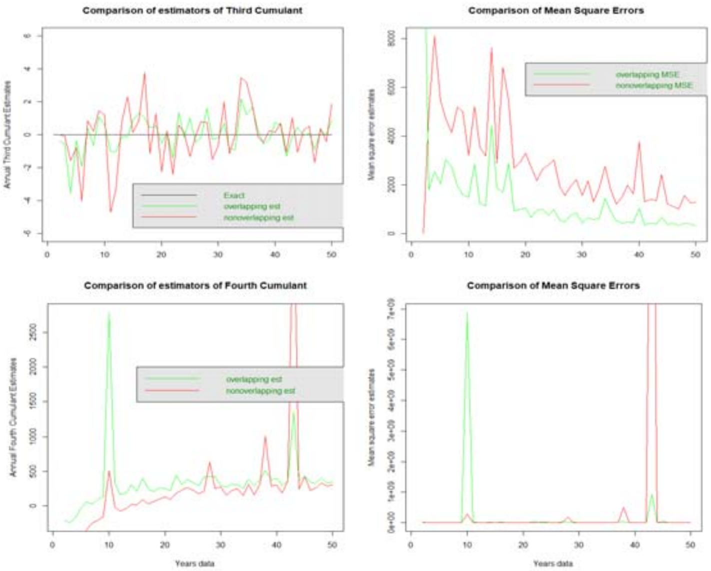
Simulation Study Results – GARCH 3rd / 4th cumulants



- 3rd cumulant:
  - Non-overlapping data has higher bias and MSE compared to overlapping data
- 4th cumulant:
  - Non-overlapping data has lower bias compared to overlapping data
  - Overlapping data has lower MSE.

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Simulation Study – Conclusions

- The results for the uncorrelated reference models (Brownian and NIG) are similar.
  - Both approaches give biased estimates for the second cumulant (variance with divisor n)
  - Bias correction factors for the variance for non-overlapping and overlapping data exist (Nelken and Cochrane – both give identical results)
  - Overlapping estimates have lower mean square errors for all the cumulant estimates – meaning they are more likely to be closer to the correct answer
- The ARIMA model has similar results to the uncorrelated reference models, but the bias corrections are not quite as good for small sample sets.
- For the second cumulant estimates for the GARCH model, the bias corrections for overlapping data and to a less extent non-overlapping data resulted in over estimates of the variance.
  - This means that neither the standard estimates nor the "bias corrected" estimates give unbiased estimates of the reference model variance.

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Solutions & Alternatives – Annualisation Transformation

- Transform monthly non-overlapping time steps into annual non-overlapping time steps by utilising the correlation between monthly data points
- Key data steps:
  - Apply autocorrelation to generate a large number of monthly steps
  - Aggregate monthly steps to come up with annualised changes
  - Annualisation using empirical marginal distributions & empirical copula to minimise information loss

Advantages	Disadvantages
Annual estimates are constructed using non-overlapping monthly observations therefore a large number of data points available for calibration	Autocorrelation in data set creation makes classical formulas AIC, goodness-of-fit invalid
Limited loss of information during data transformations	
Provides a large sample & leads to stable results in cases where data is limited	

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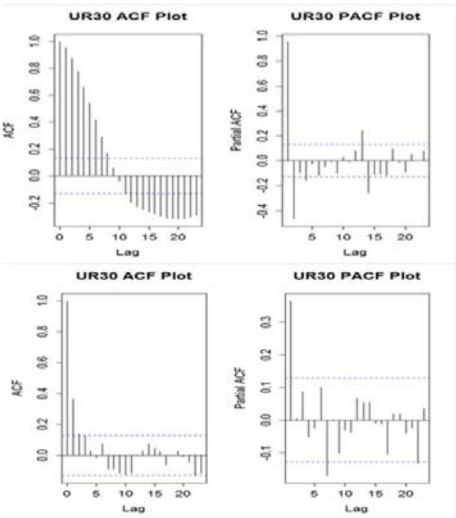
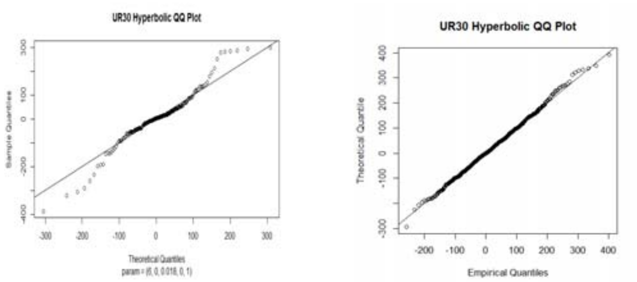
Solutions & Alternatives – Annualisation Transformation





Explanation:

- Under MAO* (top left diagram) the ACF+ starts at 1 and has exponential decay.
- Under MNO+ (bottom left diagram) the ACF quickly converges to a very low number.
- Similarly the PACF+ for MNO (bottom right) shows beyond lag 2 are within the 95% CI in comparison to MNO (top right).
- Material improvement in the fits under annualisation process

+ MAO: Monthly Annual Overlapping; +MNO: Monthly Non-Overlapping; + UR30: Merrillynch A rated index
+PACF: Partial Autocorrelation Function; +ACF: Autocorrelation Function

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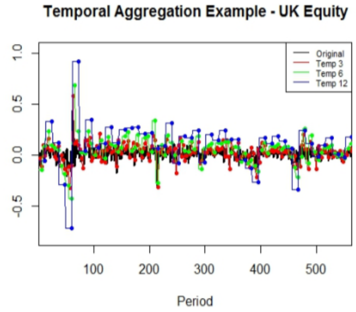
Solution – Temporal Aggregation

- Aggregates high-frequency process into low-frequency process, e.g. monthly process into annual process
- Key steps:
  - Fit a best fit time series model to disaggregated time series (e.g. monthly data)
  - Aggregate the time series
    - either by simulating from a best fit probability distribution on residuals or
    - Use (complex) temporal aggregation formulas
  - Fit a probability distribution or time series to the aggregated time series

Advantages	Disadvantages
Annual estimates are constructed using non-overlapping monthly observations therefore a large number of data points available for calibration	Leads to loss of information considering the increased number of data transformations required
Autocorrelation in the data is accounted for to make sure the estimates are valid	Comparatively more rigorous testing of the behaviour of the residuals necessary in comparison to other approaches
It can handle data with some of the known but difficult to model stylised facts such as “volatility clustering”.	Complex to understand and communicate



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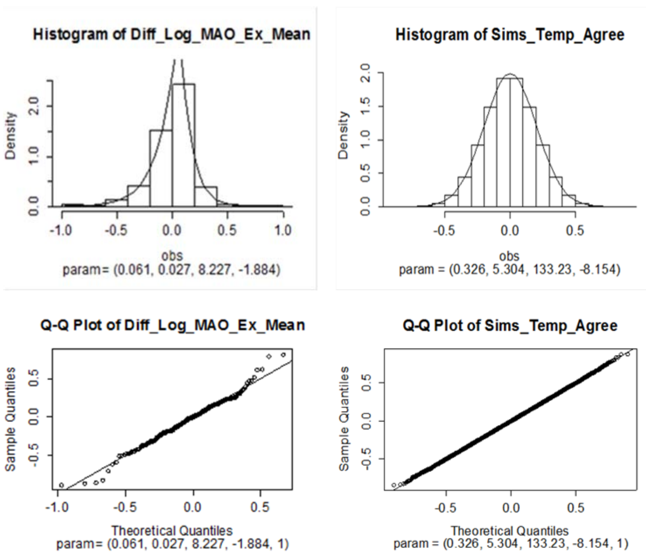
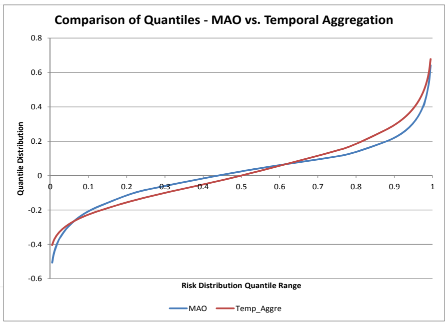
Temporal Aggregation – MAO vs. Temporal Aggregation - AR(1) Example





- Material improvement in the fits under temporarily aggregated time series
- Underestimates the extreme downside stresses

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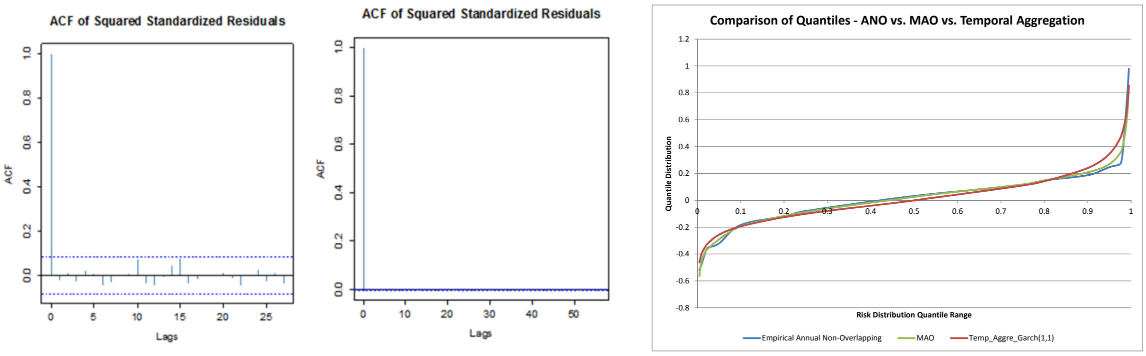
Temporal Aggregation – MAO vs. Temporal Aggregation – Garch (1,1) Example – Empiricals only



- Material improvement in the fits under temporarily aggregated Garch (1,1) process
- On the extreme, temporally aggregated Garch (1,1) process leads to stronger quantiles
- In the “body” , temporally aggregated Garch (1,1) process leads to weaker quantiles

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Summary

- Use of overlapping data is virtually the market practice despite its technical issues.
- Simulation study has shown that:
  - ***Bias:***
    - Both overlapping and non-overlapping approaches can lead to bias in the cumulants and
    - It is generally higher for overlapping data as compared to non-overlapping data
    - 2^{nd} cumulant corrections do help in removing these biases but generally at the cost of increased variance
  - ***Mean Squared Error (MSE):***
    - MSE is lower for overlapping data in comparison to non-overlapping data
- Possible solutions considered include:
  - ***Annualisation transformation:***
    - Leads to material improvement in fits
    - Introduces uncertainty due to the loss of information during data transformation and
    - It does not remove impact of autocorrelation; and
  - ***Temporal aggregation:***
    - Leads to material improvement in fits but stresses at extreme percentiles stronger compared to overlapping approach
    - Comparatively more loss of information during the multiple data transformations required
    - Complex to understand and communicate

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Questions **Comments**

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