Institute
and Faculty
of Actuaries

Gaussian emergence for stochastic reserve risk modelling

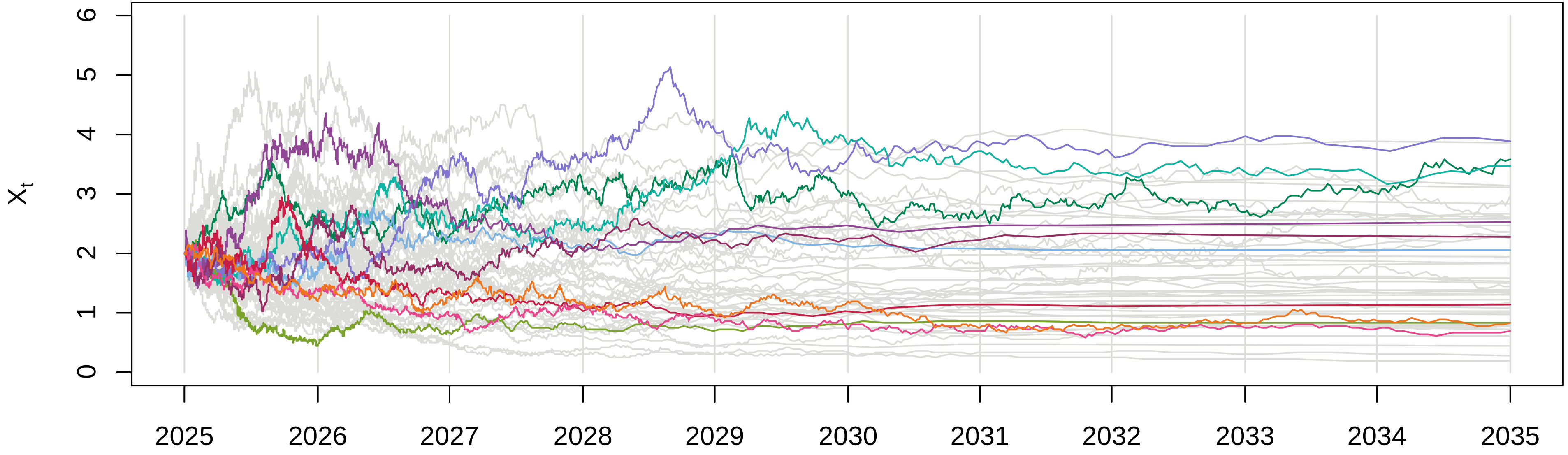
Ramsay Nashef

23 April 2025

Welcome

Disclaimers

1. The theory presented here has not been published or formally reviewed.
2. It is not necessarily a new invention.



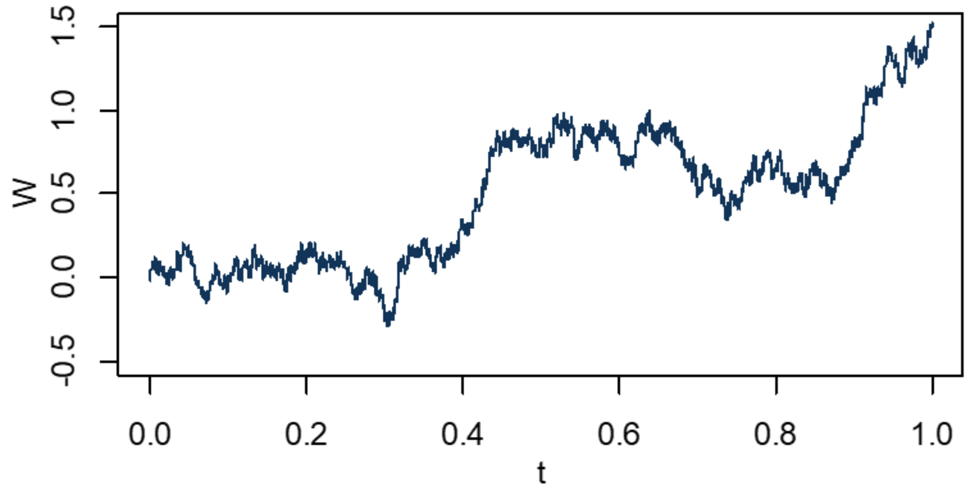
Institute and Faculty of Actuaries

Prior knowledge

Required

A **Brownian motion** (aka Wiener process) is:

- a random process (W_t) beginning at $W_0 = 0$
- with continuous sample paths
- and independent, Gaussian increments.

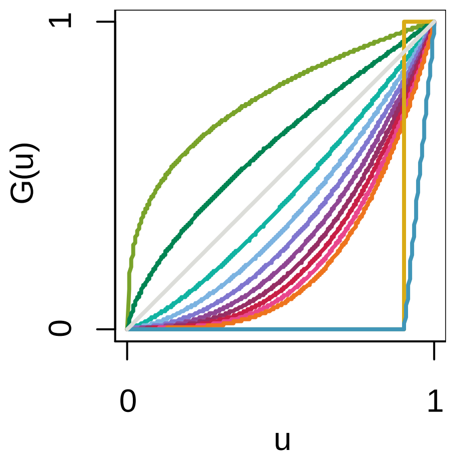


Optional

A **distortion function** is:

- a non-decreasing function
- mapping $[0,1]$ **onto** $[0,1]$.

It can be interpreted as a change of probability measure, re-weighting the likelihood of different scenarios.



Institute and Faculty of Actuaries

What are we trying to solve?

An insurer's balance sheet includes liabilities for the expected cost of unknown future risk.

Suppose we have already modelled the potential ultimate cost of risk, calculated the expectation and reserved a best estimate liability for this risk.

Next we ask, how could this estimate of the risk develop over time?

- distribution of potential future estimates for the risk
- capital requirements over different time horizons
- future capital requirements conditional on future scenarios
- expected cost of capital required over lifetime of risk (risk margin).

Table 1. Acme Inc. balance sheet as at 31/12/2024

Assets		Liabilities	
Cash	£2400	Expected cost of claims	£2000
Inventory	£1600	Equity	£2000
Total	**£4000**	**Total**	**£4000**

Institute and Faculty of Actuaries

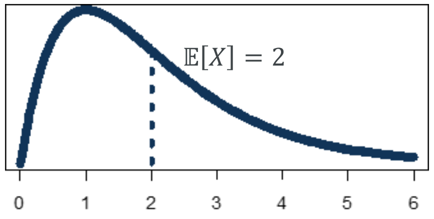
How should an emergence model behave?

Let X represent the ultimate losses from a particular risk.

Let $T = [0,1]$ represent time.

Let $X_t = \mathbb{E}_t[X]$ represent the expectation of X conditional on information known at time t.

Example risk: $X \sim Gamma(2)$



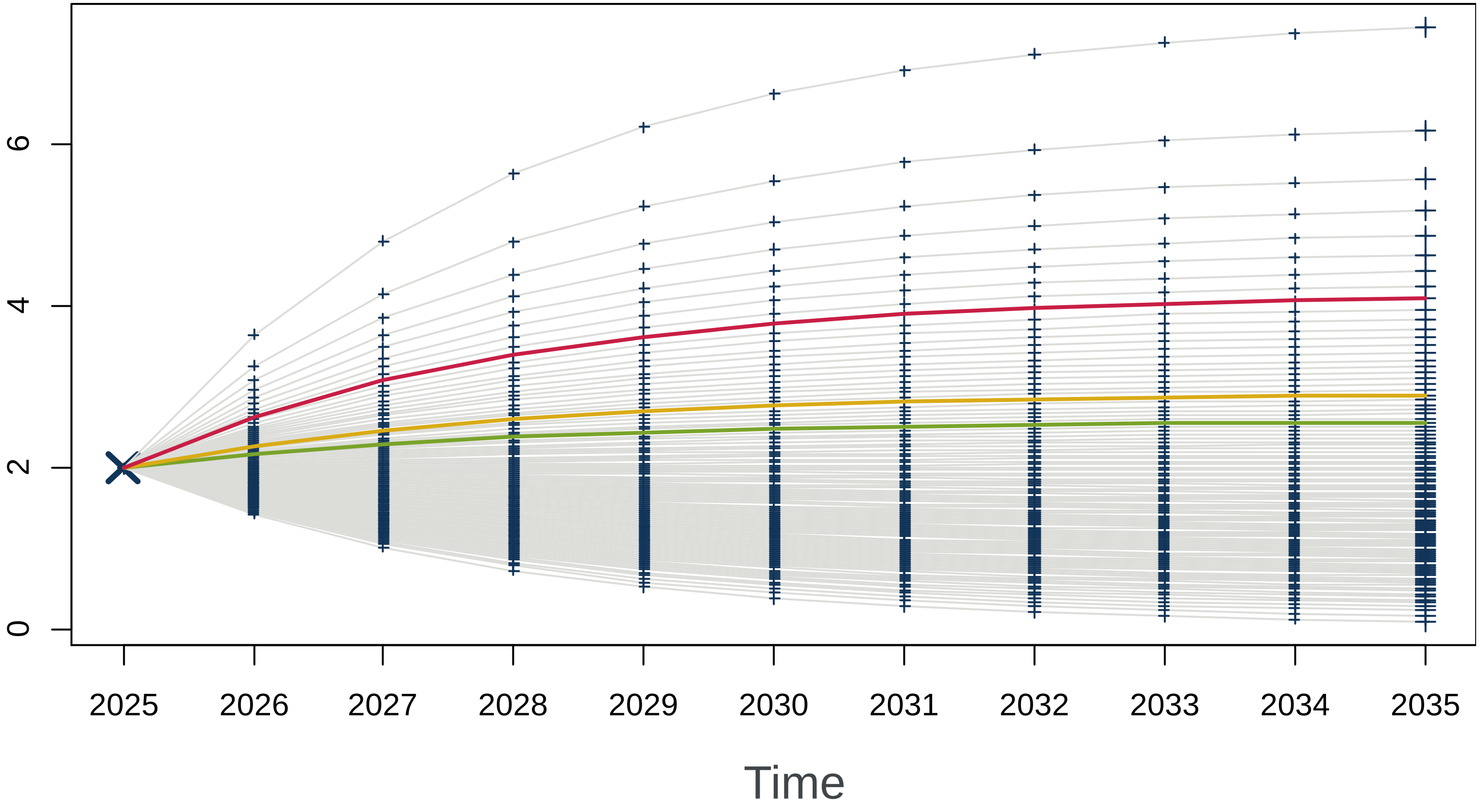
Requirements for the random process (X_t):

- begin at mean: $X_0 = \mathbb{E}[X]$
- end at ultimate: $X_1 = X$
- be a martingale: if $s < t$, then $\mathbb{E}[X_t | X_s] = X_s$.

Institute and Faculty of Actuaries

Standard approach

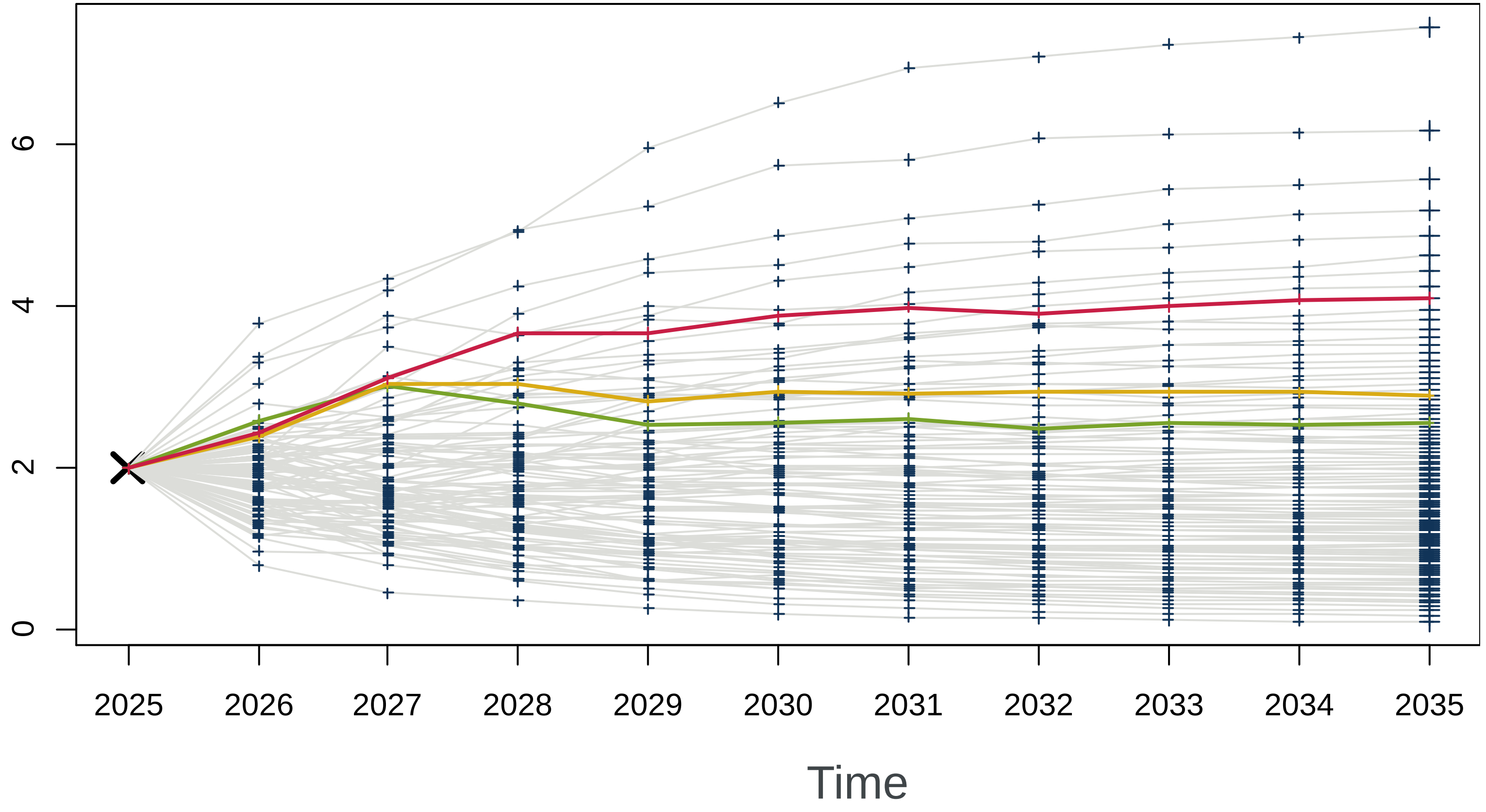
Straight-line emergence



Institute and Faculty of Actuaries

New approach

Gaussian emergence



Institute and Faculty of Actuaries

Gaussian emergence

The equation for Gaussian emergence is $\boldsymbol{F(X) = \Phi(W_1)}$, where:

- $F(\,)$ is the distribution function of the ultimate risk X
- $\Phi(\,)$ is the distribution function of the standard Normal distribution
- W_t is a standard Brownian motion, **which we observe** over time $t \in [0,1]$.

This results in the following:

$$X_t = \mathbb{E}[X|W_t] = \mathbb{E}\left[F^{-1}\left(\Phi\left(W_t + \mathcal{N}(0,1-t)\right)\right)\middle|W_t\right].$$

“given W_t” means W_t has been observed

Convert from distribution of W_1 to distribution of X

Cumulative risk emerged at time t

A normal random variable containing the remaining risk to time 1

Institute and Faculty of Actuaries

Approximate calculation with nested simulations

$$X_t = \mathbb{E}[X|W_t] = \mathbb{E}\left[F^{-1}\left(\Phi\left(W_t + \mathcal{N}(0,1-t)\right)\right)\middle|W_t\right]$$

We can easily approximate X_t using nested simulations:

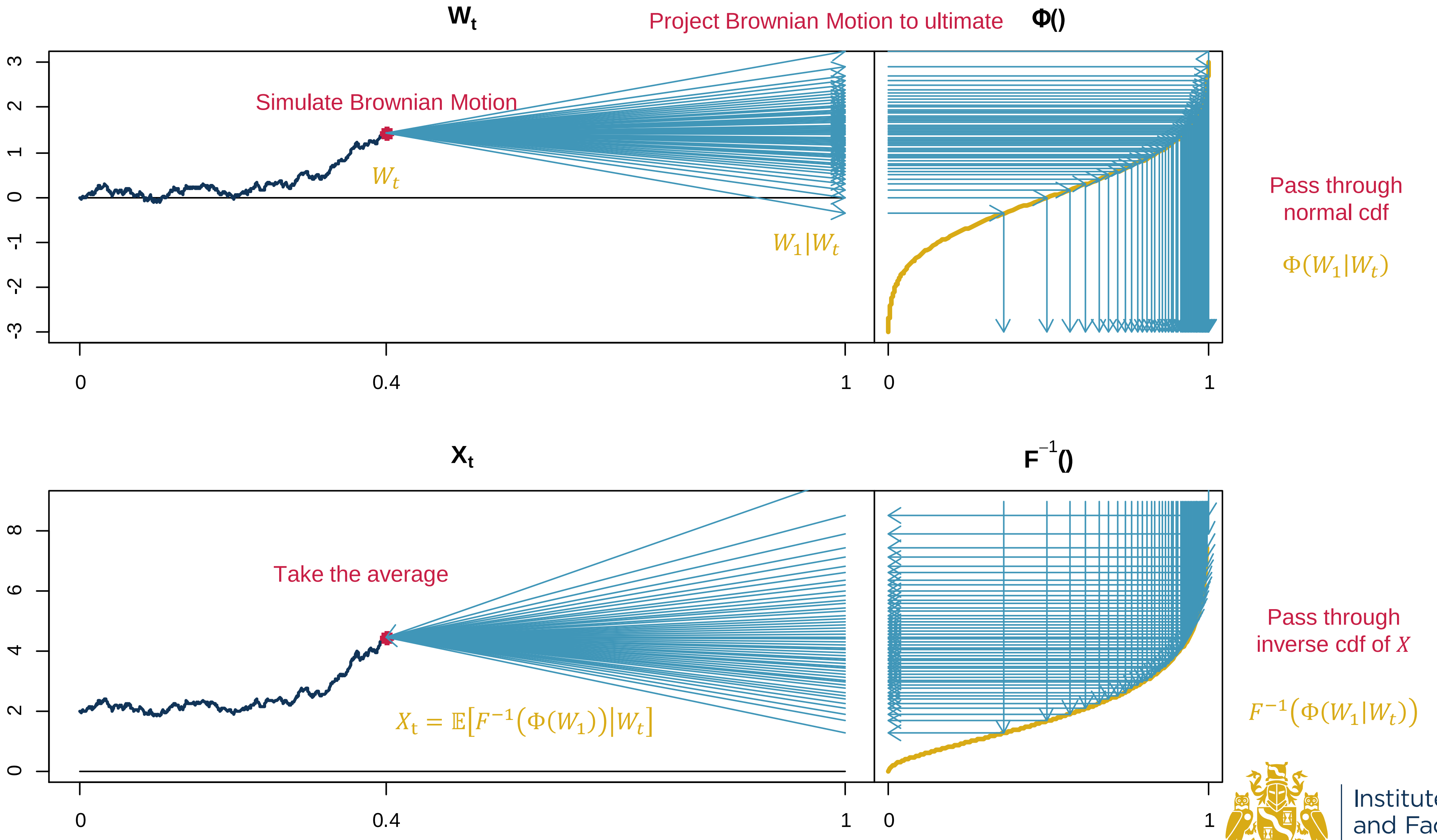
$X_t \approx \frac{1}{M}\sum_{i=1}^{M} F^{-1}\left(\Phi\left(W_t + \sqrt{1-t}\Phi^{-1}(u_i)\right)\right)$, where u_i are a sample of $\mathcal{U}(0,1)$, such as $u_i = \frac{i-½}{M}$.

- In simulation models, $F^{-1}(\,)$ can be calculated by looking up values on the distribution of X.
- M does not need to be huge.
  - $M = 100$ seems enough for accuracy within about 1%.

Institute and Faculty of Actuaries

Visual

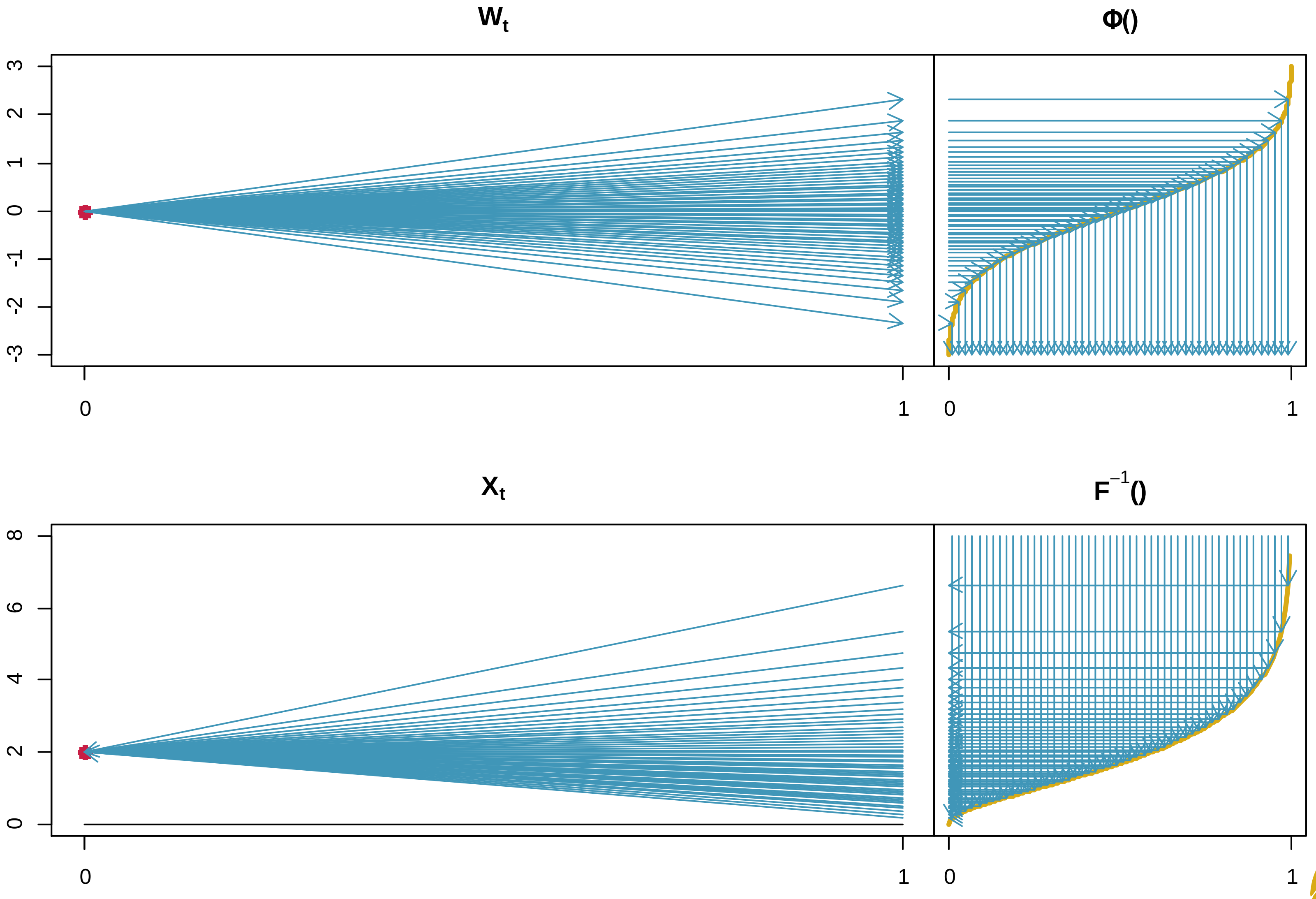
$t = 0.4$



Institute and Faculty of Actuaries

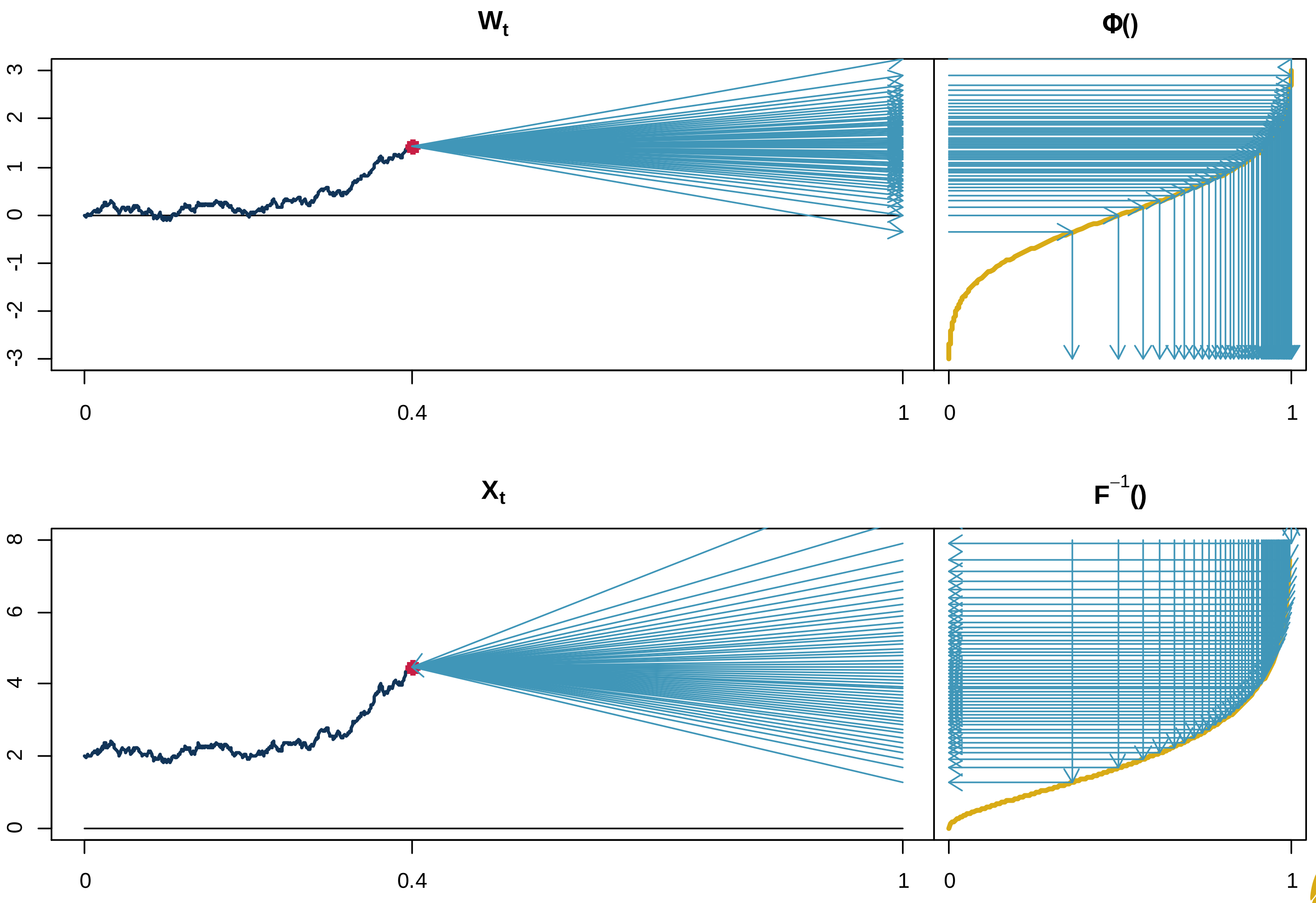
Visual

$t = 0.0$



Visual

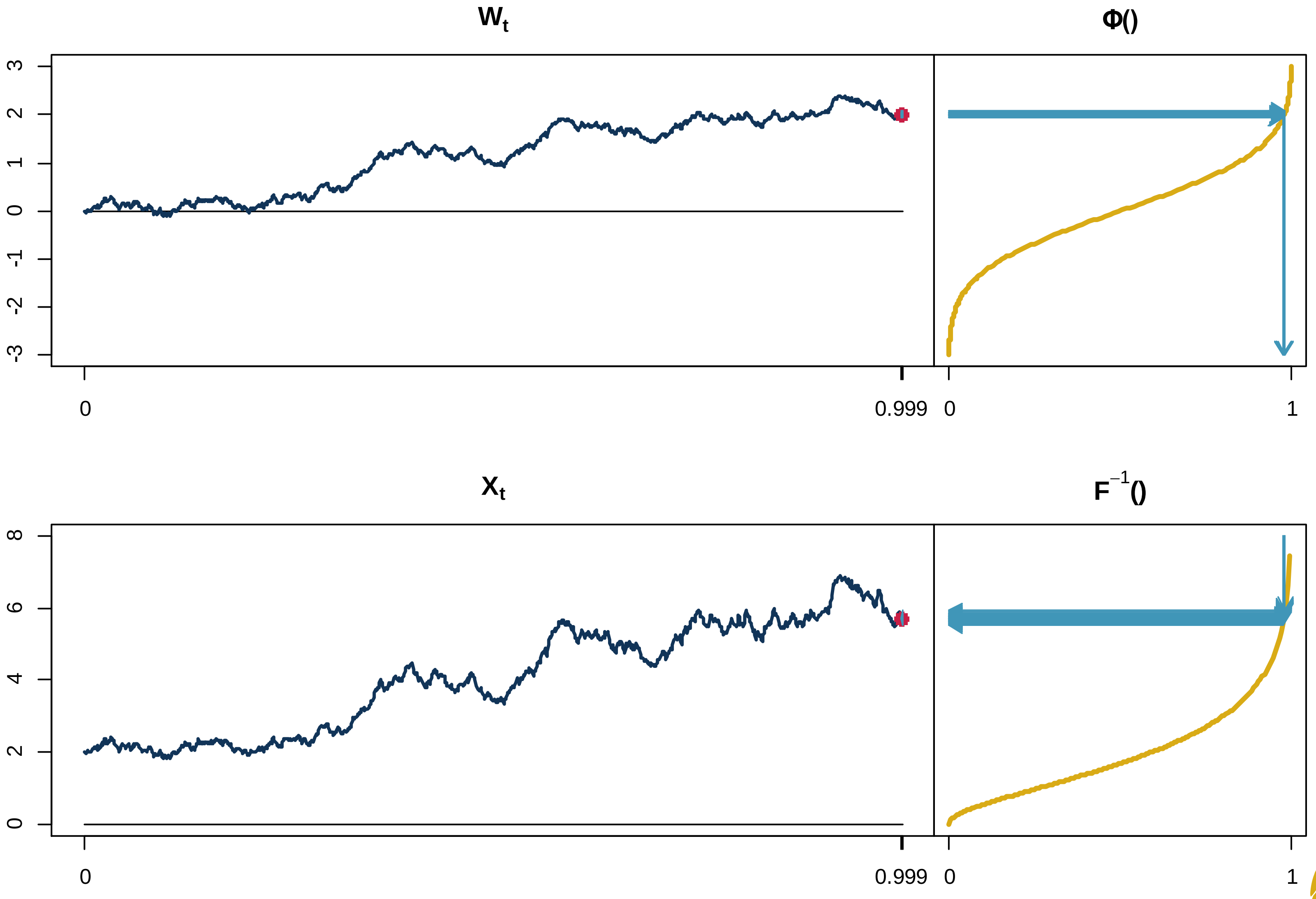
$t = 0.4$



Institute and Faculty of Actuaries

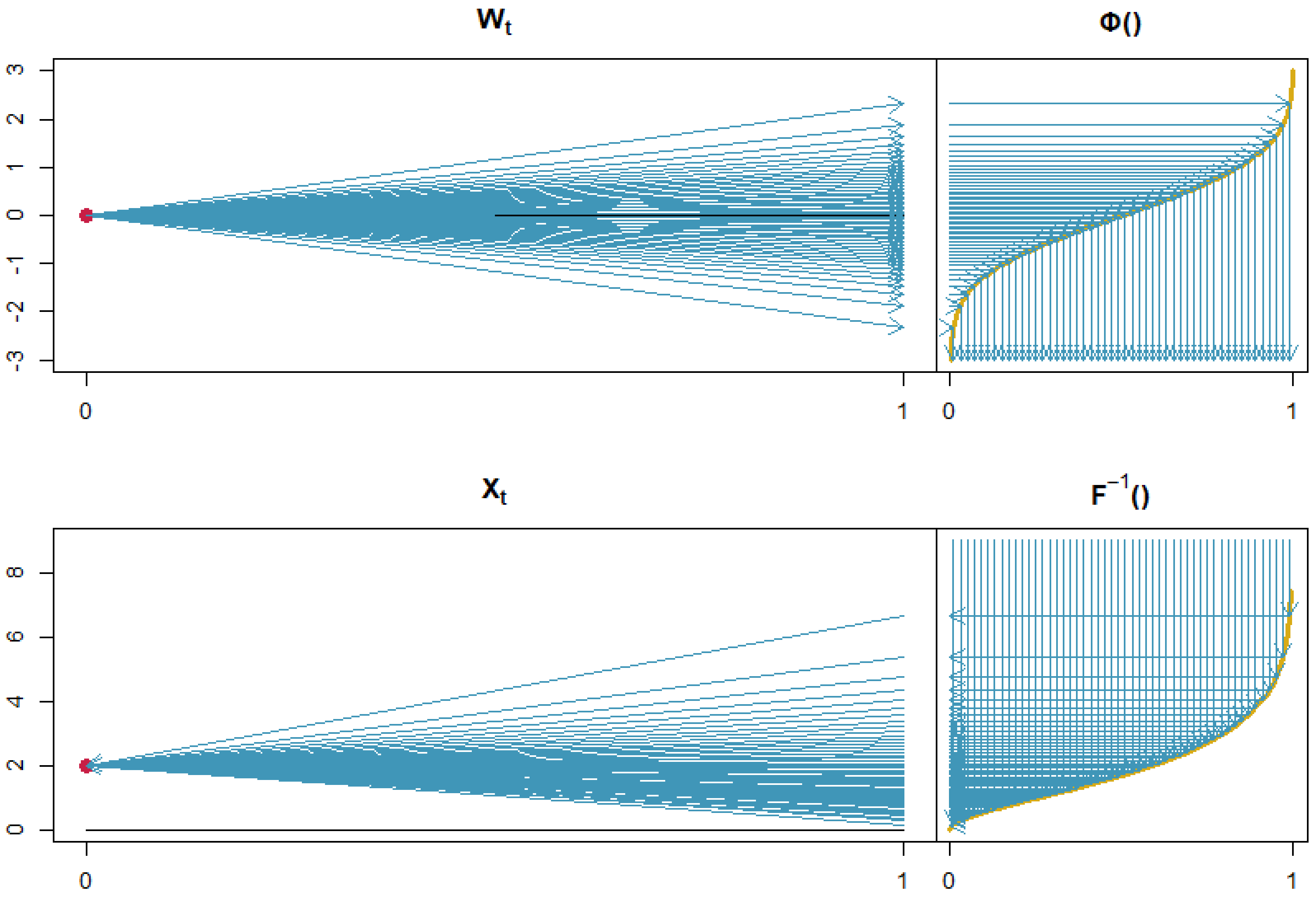
Visual

$t = 0.999$



Institute and Faculty of Actuaries

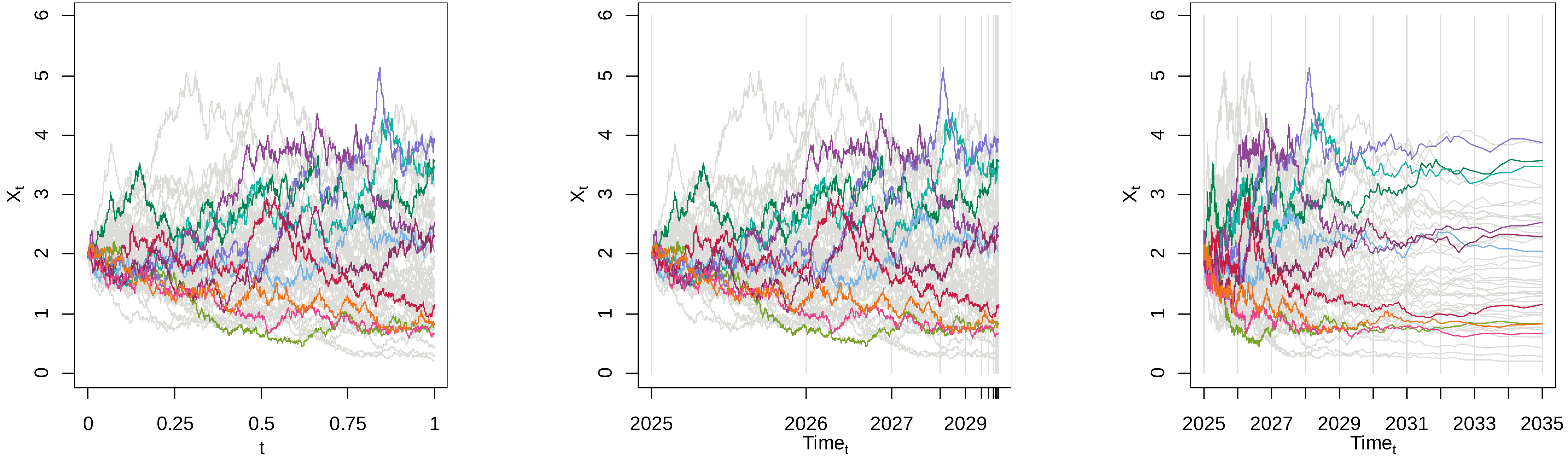
Visual
GIF



Institute and Faculty of Actuaries

Emergence patterns (deterministic)

The rate of emergence is controlled by adjusting the time axis, mapping real-life time into $[0,1]$.



Institute and Faculty of Actuaries

Key features

- Regardless of the distribution of X, the method guarantees that X_t begins at the mean of X, ends at X, and is a martingale.
- It's fairly efficient to approximate X_t.
- We can control the rate of emergence by controlling the rate that t moves from 0 to 1.

Bonus feature: If X is (Log)Normal, then X_t is naturally a (Geometric) Brownian motion:

  - if $X \sim \mathcal{N}(\mu, \sigma)$ then $X_t = \mu + \sigma W_t$
  - if $\ln(X) \sim \mathcal{N}(\mu, \sigma)$ then $\ln(X_t) = \mu + \sigma W_t + ½(1 - t)\sigma^2$.

Open question: Are there analytical solutions for other special cases?

Institute and Faculty of Actuaries

Limitation #1 – fixed rates of risk emergence

The Gaussian emergence model is overly simplistic if you assume that a known amount of risk from the Brownian motion emerges in each time period.

This does not reflect real life, where the information about risks may emerge at unknown rates.

Solution

The emergence rate can easily be made stochastic. Instead of mapping each future time to a fixed $t \in [0,1]$, you use something random.

Examples: Dirichlet allocation, Poisson process.

This does not create major difficulties in stochastic simulation of the reserve process.

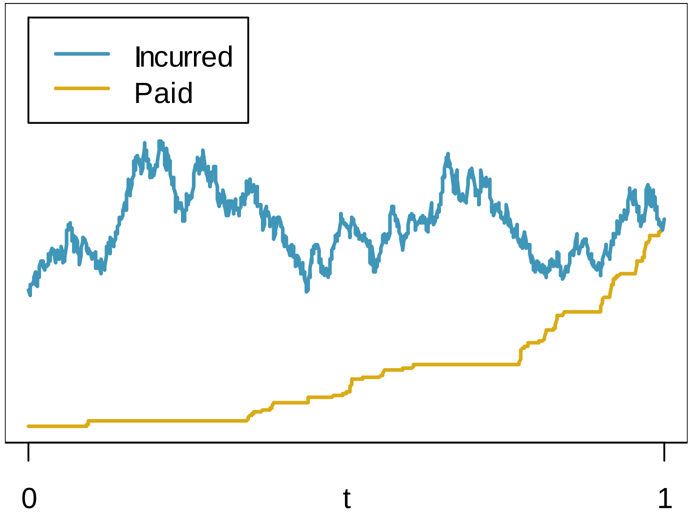
However, it makes some of the maths more difficult. In particular, Value-at-Risk calculations are more complicated because the amount of "emergence time" in a time period is uncertain.

Institute and Faculty of Actuaries

Limitation #2 – no concept of payment timing

In real-life reserve risk settings, there is usually a Paid amount which also emerges over time and acts as a lower bound for the ultimate risk. This dynamic is not present in the model.

As a result, regardless of how much emergence time has elapsed and how high a level X_t has reached, there always remains a possibility that it subsequently drops to the lowest possible value of X.



Challenge

It could be possible to develop a similar model based on Brownian motions for the simultaneous emergence of Paid and Incurred based, as envisaged in the graph to the left.

Institute and Faculty of Actuaries

Value-at-Risk and Solvency II Risk Margin

- Let $V_{t\to h}(p)$ be the Value-at-Risk of X_t over horizon $h \le 1-t$ at probability level p, satisfying: $\mathbb{P}[X_{t+h} \le V_{t\to h}(p) | W_t] = p$.
- This is calculated just as easily as X_t:

$$V_{t\to h}(p) = \mathbb{E}\left[F^{-1}\left(\Phi\left(W_t + \boldsymbol{\sqrt{h}\,\Phi^{-1}(p)} + \sqrt{1-t-\boldsymbol{h}}\,\Phi^{-1}(U)\right)\right)\middle| W_t\right], \qquad U \sim U(0,1).$$

- For fixed horizon h and level p, $V_{t\to h}(p)$ is also a martingale.
- This tells us that $\mathbb{E}[V_{t\to h}(p)] = V_{0\to h}(p)$, enabling a calculation of the Solvency II Risk Margin as the <u>expected</u> discounted cost of capital:

$$\text{Risk Margin} := C \sum_{i=0}^{N-1} \text{DiscountFactor}_i \times \mathbb{E}\left[V_{t_i \to (t_{i+1}-t_i)}(99.5\%) - X_{t_i}\right]$$

$$\approx C \sum_{i=0}^{N-1} \frac{\text{DiscountFactor}_i}{M}\left(\sum_{k=1}^{M} F^{-1}\left(\Phi\left(\sqrt{t_{i+1}-t_i}\,\Phi^{-1}(99.5\%) + \sqrt{1-(t_{i+1}-t_i)}\,\Phi^{-1}(u_k)\right)\right) - \mathbb{E}[X]\right).$$

where C is the assumed cost of capital and (t_i) are future calendar years mapped to $[0,1]$.

Note that Limitation #1 suggests that this calculation is imprudent!

Institute and Faculty of Actuaries

Emergence by distortion

When we observe W_t we change our view of the likely ultimate W_1.

This is the same as re-weighting the probabilities of W_1 with a distortion function $G_t(u)$:

- $G_t(u) := \Phi\left(\frac{\Phi^{-1}(u)-W_t}{\sqrt{1-t}}\right)$
- $G_t^{-1}(u) = \Phi\left(W_t + \sqrt{1-t}\,\Phi^{-1}(u)\right)$.

The distortion provides an updated view of the distribution of W_1 and therefore of X.

X_t is simply the **distorted expectation** of X using this distortion function:

- Expectation: $\mathbb{E}[X] = \int x f(x) dx = \int_0^1 x dF(x) = \int_0^1 F^{-1}(u) du \approx \frac{1}{M} \Sigma_{i=1}^{M} F^{-1}(u_i)$
- Distorted expectation: $\mathbb{E}_t[X] = \int_0^1 x dG_t\big(F(x)\big) = \int_0^1 F^{-1}\Big(G_t^{-1}(u)\Big) du \approx \frac{1}{M} \Sigma_{i=1}^{M} F^{-1}\Big(G_t^{-1}(u_i)\Big)$.

Institute and Faculty of Actuaries

Making X more complicated

Suppose $X = f(X_1, X_2, \dots)$ is a complicated function of a vector of multiple random variable components X_i with distribution functions $F_i(\,)$, each with their own separate emergences $W_{i;t}$ and associated distortions $G_{i;t}(\,)$.

Then $\mathbb{E}\left[X \middle| W_{1;t_1}, W_{2;t_2}, \dots\right] = \mathbb{E}\left[f\left(F_1^{-1}\left(G_{1;t_1}^{-1}(U_1)\right), F_2^{-1}\left(G_{2;t_2}^{-1}(U_2)\right), \dots\right) \middle| W_{1;t_1}, W_{2;t_2}, \dots\right]$.

The complexity of the calculation increases in line with the number of variables and the function f, but for the approximate mean calculation, M still only needs to be large enough for a decent sample average.

Example: $X = X_A + 2X_B$.

If X_A has emerged by $t_A \in [0,1]$ and X_B by $t_B \in [0,1]$ then:

$$X_{t_A,t_B} \approx \frac{1}{M}\sum_{i=1}^{M} F_A^{-1}\left(G_{A;t_A}^{-1}(u_{A,i})\right) + 2F_B^{-1}\left(G_{B;t_B}^{-1}(u_{B,i})\right).$$

If (X_i) are correlated, the dependency develops naturally over the course of emergence.

Recap

- Framing of the problem:
  - We start off already having a model of the ultimate risk X.
  - We want to find a suitable process (X_t) representing the best estimate as risk emerges.
- Solution to use a Brownian motion $(W_t)_{t\in[0,1]}$ as a proxy for the emergence of X.
- The key equation is $\boldsymbol{F(X) = \Phi(W_1)}$, which equates the rank of X to the terminal rank of (W_t), which we will observe.
- Future reserve paths can be simulated with a fairly quick approximation, outlined below:

Observe W_t → Project a sample of $W_1|W_t$ → Calculate $F^{-1}\left(\Phi(W_1|W_t)\right)$ → Average across the sample

- We can control the rate of emergence by controlling how real-life time maps into $T = [0,1]$.
- Limitations include:
  - Deterministic emergence rates will generally understate the volatility.
    - Stochastic emergence patterns solve this but make the maths more complex.
  - There is no concept of payment timing in the model.

Institute and Faculty of Actuaries

Questions

Comments

Expressions of individual views by members of the Institute and Faculty of Actuaries and its staff are encouraged.

The views expressed in this presentation are those of the presenter.

Institute and Faculty of Actuaries