Gaussian emergence for stochastic reserve risk modelling - Accompanying note for IFoA webinar - 23rd April 2025 - Ramsay Nashef

Disclaimer – the methodology outlined here is not yet published or formally reviewed.

The aim is to model the emergence of a random variable X by producing a martingale process $(X_t)_{t\in[0,1]}$ with $X_0 = \mathbb{E}[X]$ and $X_1 = X$. Loosely speaking, X_t represents the mean best estimate of X at a future time when the proportion of the total information that has emerged about X is t.

Let X be an absolutely continuous random variable with distribution function F. Let $(W_t)_{t\in[0,1]}$ be a Brownian motion, $(\mathcal{F}_t)_{t\in[0,1]}$ the corresponding natural filtration, and Φ the normal distribution function.

The Gaussian emergence equation is $F(X) = \Phi(W_1)$, which equates the rank of X to the rank of the terminal value of the Brownian motion. Thus, observing W_t over time provides information about X and yields a range of stochastic processes related to X. We can define the **median process** as M_t such that $\mathbb{P}[X \le M_t | W_t] = ½$. It is easy to see that $M_t = F^{-1}\big(\Phi(W_t)\big)$. Similarly, we can define the **quantile process** at probability level p as the ultimate Value-at-Risk ${}_pV_t$ satisfying $\mathbb{P}\big[X \le {}_pV_t \big| W_t\big] = p$. It is easy to see that ${}_pV_t = F^{-1}\left(\Phi\left(W_t + \sqrt{1-t}\,\Phi^{-1}(p)\right)\right)$.

We are particularly interested in the **mean process**, defined as $X_t \coloneqq \mathbb{E}[X|\mathcal{F}_t]$. Clearly, $X_0 = \mathbb{E}[X]$ and $X_1 = X$. Furthermore, X_t is a martingale, which follows from the tower property of conditional expectation: for $0 \le s \le t \le 1$, $\mathbb{E}[X_t|\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[X|\mathcal{F}_t]|\mathcal{F}_s] = \mathbb{E}[X|\mathcal{F}_s] = X_s$.

We can rewrite $X_t = \mathbb{E}\big[F^{-1}\big(\Phi(W_1)\big)\big|W_t\big] = \mathbb{E}\left[F^{-1}\left(\Phi\big(W_t + (W_1 - W_t)\big)\right)\middle|W_t\right]$. We know that $W_1 - W_t \sim \mathcal{N}(0, 1-t)$ is independent of W_t and is equal in distribution to $\sqrt{1-t}\,\Phi^{-1}\big(\mathcal{U}(0,1)\big)$. Therefore, $X_t = \mathbb{E}\left[F^{-1}\left(\Phi\left(W_t + \sqrt{1-t}\,\Phi^{-1}(U)\right)\right)\middle|W_t\right]$ where $U \sim \mathcal{U}(0,1)$ is independent of W_t. This can be expressed as an integral over the (uniformly distributed) possible values of U:

$X_t = \int_0^1 F^{-1}\left(\Phi\left(W_t + \sqrt{1-t}\,\Phi^{-1}(u)\right)\right) du$. Since F^{-1} and Φ are strictly increasing functions, we see that X_t is a strictly increasing function of W_t and so W_t and X_t are co-monotonic.

For financial risk management, we are interested in potential movements in the mean liability, so we define the **shock process** at probability level p over horizon $h \in [0, 1-t]$ as the Value-at-Risk ${}_pV_{t\to h}$ which satisfies $\mathbb{P}\big[X_{t+h} \le {}_pV_{t\to h}\big|W_t\big] = p$.

Since X_{t+h} is a strictly increasing function of W_{t+h}, we have that for $w \in \mathbb{R}$, conditional on W_t, $W_{t+h} \le w \Leftrightarrow X_{t+h} \le \mathbb{E}\left[F^{-1}\left(\Phi\left(w + \sqrt{1-t-h}\,\Phi^{-1}(U)\right)\right)\middle|W_t\right]$. We know that $\mathbb{P}\big[W_{t+h} \le W_t + \sqrt{h}\,\Phi^{-1}(p)\big|W_t\big] = p$. Therefore, $\mathbb{P}\left[X_{t+h} \le \mathbb{E}\left[F^{-1}\left(\Phi\left(W_t + \sqrt{h}\,\Phi^{-1}(p) + \sqrt{1-t-h}\,\Phi^{-1}(U)\right)\right)\right]\middle|W_t\right] = p$, so:

$$\begin{aligned}{}_pV_{t\to h} &= \mathbb{E}\left[F^{-1}\left(\Phi\left(W_t + \sqrt{h}\,\Phi^{-1}(p) + \sqrt{1-t-h}\,\Phi^{-1}(U)\right)\right)\middle|W_t\right]\\ &= \int_0^1 F^{-1}\left(\Phi\left(W_t + \sqrt{h}\,\Phi^{-1}(p) + \sqrt{1-t-h}\,\Phi^{-1}(u)\right)\right) du.\end{aligned}$$

Intuitively, ${}_pV_{t\to h}$ is the mean you would get after W_t undergoes a shock at probability level p.

We can see that ${}_pV_{t\to h}$ is also a martingale, i.e. for $0 \le s \le t \le 1-h$, $\mathbb{E}\left[{}_pV_{t\to h}\middle|\mathcal{F}_s\right] = {}_pV_{s\to h}$, by first rewriting ${}_pV_{t\to h} = \mathbb{E}\left[F^{-1}\left(\Phi\left(W_s + (W_t - W_s) + \sqrt{h}\,\Phi^{-1}(p) + \sqrt{1-t-h}\,\Phi^{-1}(U)\right)\right)\middle|W_t\right] = \mathbb{E}\left[F^{-1}\left(\Phi\left(W_s + (W_t - W_s) + \sqrt{h}\,\Phi^{-1}(p) + \sqrt{1-t-h}\,\Phi^{-1}(U)\right)\right)\middle|\mathcal{F}_t\right]$.

By the tower property of conditional expectation, $\mathbb{E}\left[{}_pV_{t\to h}\middle|\mathcal{F}_s\right] =$
$\mathbb{E}\left[\mathbb{E}\left[F^{-1}\left(\Phi\left(W_s + (W_t - W_s) + \sqrt{h}\,\Phi^{-1}(p) + \sqrt{1-t-h}\,\Phi^{-1}(U)\right)\right)\middle|\mathcal{F}_t\right]\middle|\mathcal{F}_s\right]$

$= \mathbb{E}\left[F^{-1}\left(\Phi\left(W_s + (W_t - W_s) + \sqrt{h}\,\Phi^{-1}(p) + \sqrt{1-t-h}\,\Phi^{-1}(U)\right)\right)\middle|\mathcal{F}_s\right]$.

$(W_t - W_s)$ and $\sqrt{1-t-h}\,\Phi^{-1}(U)$ are normal variables with variances $(t-s)$ and $(1-t-h)$, independent of each other and of $\mathcal{F}_s$. Their sum is therefore also independent of $\mathcal{F}_s$ and equal in distribution to a normal variable with variance $(1-s-h)$, which is equal in distribution to $\sqrt{1-s-h}\,\Phi^{-1}(U')$ where $U' \sim \mathcal{U}(0,1)$. Therefore:

$$\begin{aligned}\mathbb{E}\left[{}_pV_{t\to h}\middle|\mathcal{F}_s\right] &= \mathbb{E}\left[F^{-1}\left(\Phi\left(W_s + \sqrt{h}\,\Phi^{-1}(p) + \sqrt{1-s-h}\,\Phi^{-1}(U')\right)\right)\middle|\mathcal{F}_s\right]\\ &= \mathbb{E}\left[F^{-1}\left(\Phi\left(W_s + \sqrt{h}\,\Phi^{-1}(p) + \sqrt{1-s-h}\,\Phi^{-1}(U')\right)\right)\middle|W_s\right] = {}_pV_{s\to h}.\end{aligned}$$

Finally, we may define the **capital requirement process** as ${}_pC_{t\to h} = {}_pV_{t\to h} - X_t$, which represents the risk over horizon h of an increase in the best estimate liability at a probability level p. Since both parts are martingales, this too is a martingale. Informally, this tells us that under Gaussian emergence, the expected future reserve risk over a fixed horizon h is equal to the present reserve risk over that horizon – even though the future capital requirements are uncertain and will depend on how the risk develops. The calculation of the risk margin in Solvency II implicitly relies on expected future capital requirements (although the regulation does not explicitly mention that future capital requirements are themselves random variables).

A word of caution: in real life, information does not emerge at predictable rates. There are two drivers of risk emergence: (1) How much information will emerge? and (2) Will the information be good or bad? Gaussian emergence only models (2); the rate of information emergence is left to the user. We might assume that 50% of the risk will emerge in the first year, 30% in the second, and the remaining 20% in the third. Then today's estimate is $X_0 = \mathbb{E}[X]$, next year's is modelled as $X_{0.5}$, the next as $X_{0.8}$, and the process finishes after three years at $X = X_1$. However, insurance reserves depend on informative events such as claim notifications, settlements and court cases, the timing of which cannot be predicted with certainty. Assuming no variance in the amount of "emergence time" elapsing in each time period will generally underestimate the volatility of the process and understate the capital requirements and the risk margin.

This is easily overcome by simulating stochastic rates of emergence instead. For example, using Poisson processes or Dirichlet allocation. But this presents challenges. In particular, the shock process defined above is defined in terms of a fixed horizon of "emergence time", rather than real-life time. It is more complex to calculate the Value-at-Risk when the amount of "emergence time" in the real-life time period is uncertain. Furthermore, there remains the question of whether the amount of information that has emerged about the risk is known during the process. More work is required in this area.