Communication and Personal Selection of Pension Saver's Financial Risk

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Abstract

The paper shows how to reform the platform of pension products so that pension savers, professional financial advisors, actuaries and investment experts intuitively understand the underlying financial risk of the optimal investment profile. It is also pointed out that an excellent optimal investment strategy can destroy the future expected utility of a pension saver if the financial communication is wrong. It is shown that a simple system with an upper and a lower bound, originally inspired by Merton [Harvard Business Review, 2014, 92 (7/8), 43–50], which can be executed easily using fintech, can replace complicated power utility optimization for the pension saver so that everyone can exactly understand the amount of financial risk taken. The paper focuses on investing money as a lump sum because being able to communicate the associated financial risk can serve as the first step towards communicating more complex pension saving structures.

Keywords: investment analysis, finance, utility theory, risk management, OR in banking

1. Introduction

Communication and transparency have long been the insoluble challenge in the pension market. The Transparency Task Force was, for example, set up by the British government in connection with the recent Pension Schemes Act 2017. When the task force met in the House of Commons in November 2016, Henry Tapper, actuary and founder of work place pension broker Pension PlayPen, argued about the slow-pace development of transparency in products as well as in communications. On the same note, in summer 2016, the financial journalist Joe McGrath wrote an article in the online magazine *Raconteur* spelling out that "Pension fund trustees are under increasing pressure from members, industry leaders and regulators to achieve transparency, good returns and lower costs." These are just two examples that clearly indicate an ongoing daily effort to simplify and improve the pension system. The ultimate goal is that pension savers know exactly what they buy and how much they have to pay for it.

According to Collins (2012), the majority of the population does not understand financial risk well. It is important to ensure that the pension savers invest according to their risk preferences. Furthermore, good communication is not sufficient. Pension products have to adapt

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to simple communication until a one-one relationship between communication and financial construction is eventually reached. Simple, transparent and efficient pension platforms have to be built and such fit well into current efforts for technological innovation in the financial sector. Our starting point is the pension vision of the Nobel laureate Robert C. Merton (Merton, 2014), who argues that a new pension system must be built so that more informed decisions are made by all pension savers, including those lacking financial literacy. He suggests a pension system containing a top and a bottom rate; the pension saver reveals his risk appetite indirectly by choosing an individual combination of a top and a bottom rate according to his individual circumstances. He also argues for a clever default for those pension savers who simply refuse to take decisions. Merton did not publish technical details or communication details to his vision. This paper does exactly that: we provide in an unprecedented way a communication platform which is well-suited for an implementation by software and modern technology. We consider here the lump-sum case. While interesting in itself, it serves as the building block for the construction of a variety of pension annuity products which can have many different practical features, as shown, for example, in Gerrard et al. (2018). After informally interviewing several laymen and pension savers, we decided to simplify Merton's approach by reducing the potential choices to the pension saver down to picking the worst case (WC) investment result. The choice is aided by a best case (BC) linked to every WC; the pension saver then receives BC half of the time, and an investment result between WC and BC the rest of the time. Hence, a one-one relationship between communication and financial construction is achieved. Section 2 presents how this communication can look like, while Section 6 shows in a detailed simulation study how that suffices to back-calculate an efficient investment strategy which incorporates the right risk appetite for a specific customer. We highlight that the investment strategy is determined without an abstract estimation of a risk-appetite parameter but via a simple one-one relationship between communication and financial product.

Underlying to this communication is an original unhedged investment strategy based on an exponential utility and a hedge based on the best and worst case boundaries leading to the final strategy. While the pension savers do not need to know these technical details, we have also simplified the technical communication of our investment universe. The reason for this is that, if the financial and actuarial experts have a good grip of the technical details of their product design, then such will lead to better products, communication, fewer internal errors, and lower administration costs.

The main trick to obtain this technical simplification is to use exponential utility, rather than, for example, power utility optimization. In Section 6, we show that, while the selection of the BC and WC is important for the financial utility of the pension saver, the underlying investment strategy, i.e., the choice of a concrete utility function, plays a secondary role. It just has to be risky enough to accommodate the variety of risk preferences wished for by the pension savers when they limit their risk via the BC and WC boundaries. Going one step further, we propose that picking the WC – with an automatically calculated BC that is reached 50% of the time – is enough. It is good news that the simple decision that can be taken by pension savers is also the most important one. This is the primary reason for this paper's argument that pension savers can self-select their investment strategy by answering a simple question, e.g., via a fintech platform.

The remainder of the paper is organized as follows. In Section 2, we present an introductory example of the proposed strategy with a one-one relationship between communication and financial product. Sections 3–5 are concerned with the technical details of formulating a financial market model and with finding the optimal investment strategy thereof. In Section 6, we demonstrate in a simulation study that the simple communication of Section 2 is indeed good enough to back-calculate an optimal investment strategy tailored to the individual's risk appetite. Section 7 discusses possible extension routes of our lump-sum development and Section 8 concludes the paper. Detailed derivations are deferred to the appendix.

2. Lisa, John, Susan and James self-select their risk profiles

Consider the risk taker Lisa, the moderate risk taker John, the moderately risk averse Susan and the risk averse James. Each of them wants to invest £10,000 with an investment horizon of 30 years. Table 1 summarizes their optimal strategies in a power utility world. These correspond to the different risk appetites our four protagonists have, described via the power utility parameter ρ; refer also to our parameterization of the power utility in Section 4 for more details. The optimal strategy is derived in a two-fund investment universe within the Black–Scholes model setting (see Section 3) based on an investment of a constant relative amount of the current wealth in the risky stock and the remainder in a risk-free inflation bond. The investment outcomes in this section are therefore measured in real terms, as recommended in Merton (2014) – see also Section 3.

Investor	Lisa	John	Susan	James
Risk appetite ρ (power utility parameter)	-0.25	-1	-4	-10
Percentage in stocks	75%	46%	19%	8%

Table 1: Power utility parameters of four investors and corresponding optimal strategies. The optimal strategies are derived in a Black–Scholes world where the risky asset has a yearly mean excess return of 3.4% with a standard deviation of 16%. (See Section 3 for a detailed description of the financial universe.)

The story would end here if the protagonist's risk appetite were known a priori – but it is not. We will now show how a simple question to Lisa, John, Susan and James will tell us what kind of risk they want; we will show how they can self-select their risk and thereby their entire investment strategy via a short decision process. The explosive growth of financial technology in personal finance following the internet and mobile phone revolutions (e.g., see de Reyck and Degraeve, 2003) should be exploited to further simplify the decision process. In particular, each of the protagonists could be told via a smartphone application that:

- Your investment has a BC and a WC
- You will never drop below your WC
- Half of the time you will be receiving the BC and in the other half an investment result between WC and BC

- Use a slider to see which WC suits you best. For every WC there is a link to a BC; BC increases when WC decreases.

Investor	Lisa	John	Susan	James
WC (guarantee)	£3,650	£6,500	£9,100	£9,650
BC (achieved half of the time)	£16,500	£15,200	£12,300	£11,000

Table 2: Optimal picks of worst case and best case (the goal) for the four investors, given their corresponding power utility parameters $\rho = -0.25, -1, -4, -10$. The initial wealth is £10,000 and the investment horizon is 30 years. The optimal strategy is derived in the Black–Scholes world (see also notes in Table 1).

Table 2 shows the optimal choices of Lisa, John, Susan and James. All numbers are in current values, i.e., adjusted for inflation. Lisa's median in the unhedged world, where her optimal strategy is to hold 75% in risky assets like stocks, would have been £13,496; with the new hedging strategy, Lisa's median is £16,500, i.e., this has increased by £3,004. She also has a guarantee with her WC of £3,900, as opposed to no guarantee before. The price for the increased median and the guarantee is that Lisa will not receive above her BC amount of £16,500. In an unhedged world, Lisa would have had only 31% chance to exceed £16,500. So we take away the gamble that she can achieve super returns less than one third of the time to improve her WC scenario and the return she will achieve most of the time, hence we maximize her median return.

We note that Lisa, John, Susan and James self-selected their risk profile and defined their optimal investment strategy through a simple exercise using a slider on a mobile phone application or web interface. The question they receive is directly linked to their investment without any further, possibly abstract and detached, communication of risk preferences being needed. Table 3 compares our proposed strategy with an optimal strategy in terms of certainty equivalence. The optimal strategy is infeasible as the exact utility function and its parameter are unknown. Nevertheless, the loss in terms of optimal utility by taking our simple one-step financial advice is not material, which is our key point.

Investor	Optimal strategy	Hedged strategy		
	CE	CE	WC	BC
Lisa	£12,756	£12,017	£3,650	£16,500
John	£11,643	£11,264	£6,500	£15,200
Susan	£10,627	£10,416	£9,100	£12,300
James	£10,280	£10,171	£9,650	£11,000

Table 3: Comparison of different strategies. Investors are assumed to obey a power utility with parameters $\rho = -0.25, -1, -4, -10$, respectively. Initial wealth is £10,000 and investment horizon is 30 years. We assume a Black–Scholes world (see also notes in Table 1). Certainty Equivalent (CE) is the certain amount for which the investor would exchange the uncertain terminal lump sum.

We note that the loss is less than 2% for John, Susan and James, with Lisa's being slightly higher. If we go hunting for more sophisticated strategies, the potential gain will most likely be well below 2%. Even a very sophisticated in-depth financial interview can never lead to a perfectly optimal investment strategy, hence some uncertainty about the assessment of the exact risk profile of Lisa, John, Susan and James will always exist.

Finally, we revisit the unhedged power utility world to assess the cost of misunderstandings. Table 4 shows the value of the investment for Lisa, John, Susan and James if one of them through financial assessment is mistaken for another. The worst possible misspecification is if the risk averse James is mistaken to be a risk taker like Lisa. Then James loses almost 80% of the value of his investment. Lisa would lose almost 20% of the value of her investment had she been mistaken to be a risk averse investor like James. In 5 out of the 16 cases the protagonists not only lose money compared to the optimal strategy, but would also have been better off by investing everything in the safe asset. The financial decision suggested above should be easy to include in an innovative technological learning environment; see, e.g., Levina et al. (2009) for considerations on necessary adjustments when the consumer is being addressed online.

	Lisa's plan	John's plan	Susan's plan	James' plan
Lisa's CE	£12,756	£12,326	£11,124	£10,536
John's CE	£11,023	£11,643	£11,023	£10,516
Susan's CE	**£6,156**	**£9,268**	£10,627	£10,437
James' CE	**£2,388**	**£5,958**	**£9,879**	£10,280

Table 4: Impact of miscommunication. Investors are assumed to obey a power utility with parameters $\rho = -0.25, -1, -4, -10$, respectively (see additional notes in Table 3). Boldface cases indicate plans that are less valued than the initial wealth of £10,000.

3. The model

The underlying financial model in this paper has a two-fund structure comprising a risk-free fund and a risky fund. Under the two-fund theorem, this financial structure is more general than it appears at first sight as every efficient portfolio can be reconstructed out of those two funds.

Merton (2014) argues that all calculations and financial forecasts must be in current prices. We model the risk-free fund earning an interest rate r. If we assume that the risk-free fund is constructed so that it exactly compensates price inflation without any risk, i.e., r equals inflation, then it replicates exactly what Merton looked for. But a risk-free inflation fund is not feasible in practice. It is, however, intuitively clear that if the risk-free fund approximates an inflation hedge with a low volatility, then the proposed financial model where r equals inflation approximates a real and feasible investment strategy. In the following sections all strategies are to be understood in current prices.

By way of further detail of the financial model, consider a single investor living in a Black–Scholes environment. In the period $[0, T]$, $T > 0$, he can invest in a risky fund, S_1, and a risk-free fund, S_0:

$$dS_1(t) = \mu S_1(t)dt + \sigma S_1(t)dW_t, \quad dS_0(t) = rS_0(t)dt,$$

where W is a standard Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, μ, σ, $r > 0$ are constants and $S_0(0) = S_1(0) = 1$. The information available to the investor is represented by the filtration $\mathcal{F}_t = \sigma\{W(s), s \in [0, t]\} \vee \mathcal{N}(\mathbb{P})$, $t \in [0, T]$, where $\mathcal{N}(\mathbb{P})$ denotes the collection of all $\mathbb{P}$-null sets so that the filtration obeys the *usual conditions*. We will denote by $X(t)$ the current wealth at time t, of which $\pi(t)$ is invested in the risky fund and the remainder in the risk-free fund, so that

$$\begin{aligned} dX(t) &= r\left(X(t)-\pi(t)\right)\,dt+\left(\mu\,dt+\sigma\,dW(t)\right)\pi(t) \\ &= rX(t)\,dt+\left(\theta\,dt+dW(t)\right)\sigma\pi(t), \end{aligned}$$

where $\theta = (\mu - r)/\sigma$ is the market price of risk. In addition, we define the transformed process

$$Y(t) = e^{r(T-t)}X(t),$$

so that current wealth is measured in terminal time value. We have

$$dY(t) = \sigma e^{r(T-t)}\pi(t)\left(\theta\,dt+dW(t)\right). \tag{1}$$

By definition $Y(T) = X(T)$. This will become important in the next sections where we aim to find the optimal strategy when maximizing the terminal expected utility $\mathbb{E}[U(X(T))] = \mathbb{E}[U(Y(T))]$, given a utility function U.

4. Unconstrained optimal investment strategies

4.1. Two utility function families

For a given positive wealth x, the power utility function is given by

$$U_p(x) = \frac{1}{\rho}x^{\rho}; \quad \rho \in (-\infty, 1)\setminus 0. \tag{2}$$

This is also known as isoelastic utility function being invariant to scaling, meaning that counting the investors' money in pence, pounds, or any other currency does not alter the optimal strategy. A customer obeying this utility function is risk averse as the second derivative in x is negative. One can also calculate the absolute and relative risk aversion coefficients given, respectively, by

$$A_p(x) = -\frac{U_p''(x)}{U_p'(x)} = \frac{1-\rho}{x}, \quad R_p(x) = -x\frac{U_p''(x)}{U_p'(x)} = 1-\rho,$$

which are standard risk measures for utility functions. Hence, the power utility has a constant relative risk aversion (CRRA).

The exponential utility function is given by

$$U_e(x) = -\frac{1}{\gamma}e^{-\gamma x}, \tag{3}$$

where $\gamma > 0$, meaning that the customers are assumed to be risk averse as the second derivative in x is negative. The absolute and relative risk aversion coefficients are given by

$$A_e(x) = -\frac{U_e''(x)}{U_e'(x)} = \gamma, \quad R_e(x) = -x\frac{U_e''(x)}{U_e'(x)} = \gamma x,$$

hence the exponential utility function results in constant absolute risk aversion (CARA) and increasing relative risk aversion.

In various examples from the academic literature some complicated optimization algorithms are used to find the best future investment strategy for pension savers (e.g., Yu et al., 2012), where statistical simulation is part of the numerical weaponry. Another challenge for the pension saver is when the underlying theoretical model is at such a high academic level that ordinary people are unable to understand it (e.g., Josa-Fombellida and Rincón-Zapatero, 2008, Chai et al., 2011, Sun et al., 2017). As a result, such approaches remain black boxes for the pension savers or even the professional financial advisors. When the underlying argumentation and the substance of the optimization is opaque to the those giving financial advice and those receiving it, there is a risk that unfortunate financial decisions are made even if the underlying financial expert systems are sophisticated. For example, Agnew et al. (2018), Buell et al. (2017), Lymer and Richards (1995), and Pasiouras (2018) propose different arguments of the financial importance of transparent processes between consumer and provider.

We propose to only maximize the expected utility of the terminal lump sum, $X(T)$. As we will see in the next section, the power utility function optimization leads to a CRRA investment strategy where a constant relative fraction A is invested in the risky asset; if, for example, $A = 1$, then the whole investment is made in the risky asset at any time. Instead, the exponential utility function optimization leads to a CARA strategy where a constant nominal amount C is invested in the risky asset. The latter implies that any loss from investing in the risky asset is reinvested by transferring exactly the lost amount from the safe fund to the risky fund; instead, any gain from the risky asset is transferred to the risk-free fund. Intuitively, the CARA strategy is attractive because, according to the law of large numbers, it seems clear that the investment outcome will have a lower standard deviation than CRRA.

4.2. Optimal unconstrained investment strategy: CRRA versus CARA financial optimization and transparency

In consistency with standard optimal control theory, we define the optimal value function at time t

$$V(t, y) = \sup_{\pi} \mathbb{E}[U(Y(T))|Y(t) = y, \text{ strategy } \pi \text{ is used}].$$

The Hamilton–Jacobi–Bellman equation describing the dynamics of V is given by

$$\sup_{\pi} \left\{ V_t + \theta\sigma e^{r(T-t)}\pi(t)V_y + \frac{1}{2}\sigma^2\pi(t)^2 e^{2r(T-t)}V_{yy} \right\} = 0,$$

where V_t, V_y and V_{yy} are the partial derivatives with respect to t and y (first and second order). By utilizing the first-order condition in the optimization problem above, we find that the optimal value of π is

$$\pi^*(t, y) = -\frac{\theta}{\sigma}e^{-r(T-t)}\frac{V_y}{V_{yy}},$$

and conclude that V satisfies

$$V_t - \frac{\theta^2}{2}\frac{V_y^2}{V_{yy}} = 0.$$

Subject to the boundary conditions

$$V(T,y)=\begin{cases}\frac{1}{\rho}y^{\rho} & \text{(power utility – see Equation 2)}\\ -\frac{1}{\gamma}e^{-\gamma y} & \text{(exponential utility – Equation 3)}\end{cases},$$

it is straightforward to show that

$$V(t,y)=\begin{cases}\frac{1}{\rho}y^{\rho}e^{\frac{\theta^2}{2(1-\rho)}(T-t)} & \text{(power utility)}\\ -\frac{1}{\gamma}e^{-\frac{\theta^2}{2}(T-t)-\gamma y} & \text{(exponential utility)}\end{cases}$$

yielding the optimal strategies

$$\pi^*(t,y)=\begin{cases}Ae^{-r(T-t)}y & \text{(power utility)}\\ Ce^{-r(T-t)} & \text{(exponential utility)}\end{cases}, \tag{4}$$

where

$$A=\frac{\theta}{\sigma-\rho\sigma} \tag{5}$$

and

$$C=\frac{\theta}{\sigma\gamma}. \tag{6}$$

Not surprisingly, in the exponential utility function, the optimal amount invested in the risky asset is independent of the size of the fund. Subject to optimal control, the evolution of the size of the fund is given by

$$Y^*(t)=\begin{cases}y_0e^{\left(\theta\sigma A-\frac{1}{2}\sigma^2A^2\right)t+\sigma AW(t)} & \text{(power utility),}\\ y_0+R\left(\theta t+W(t)\right) & \text{(exponential utility)}\end{cases},$$

where

$$R=C\sigma=\frac{\theta}{\gamma}. \tag{7}$$

Both optimal strategies are comprehensible. From (4), we see that in the power utility case there is a constant *relative* amount of wealth A invested in the risky asset at different points in time; exponential utility suggests that there is a constant *nominal* amount C invested in the risky asset. Nevertheless, the choices of A and C are difficult for most pension savers.

In Section 5, we will see that the problem of choosing A and C can be avoided by applying a lower and an upper bound; a choice that most pension savers – financially literate or not – are able to make. We will show that combining CARA with financial hedging leads to an attractive and transparent investment strategy that enables simple self-selection of risk.

4.3. Local approximation of power utility functions

When a pension saver has to pick between the two presented utility functions, it can be argued that a world with investors maximizing a power utility is more realistic than the exponential utility analogue. Financial economics research is concerned with the choice of utility function for use in different economic scenarios (see, among others, Kallberg and Ziemba, 1983, Levy, 1992, Abbas, 2012 and Pliskin et al., 1980). Here, we will be choosing between power and

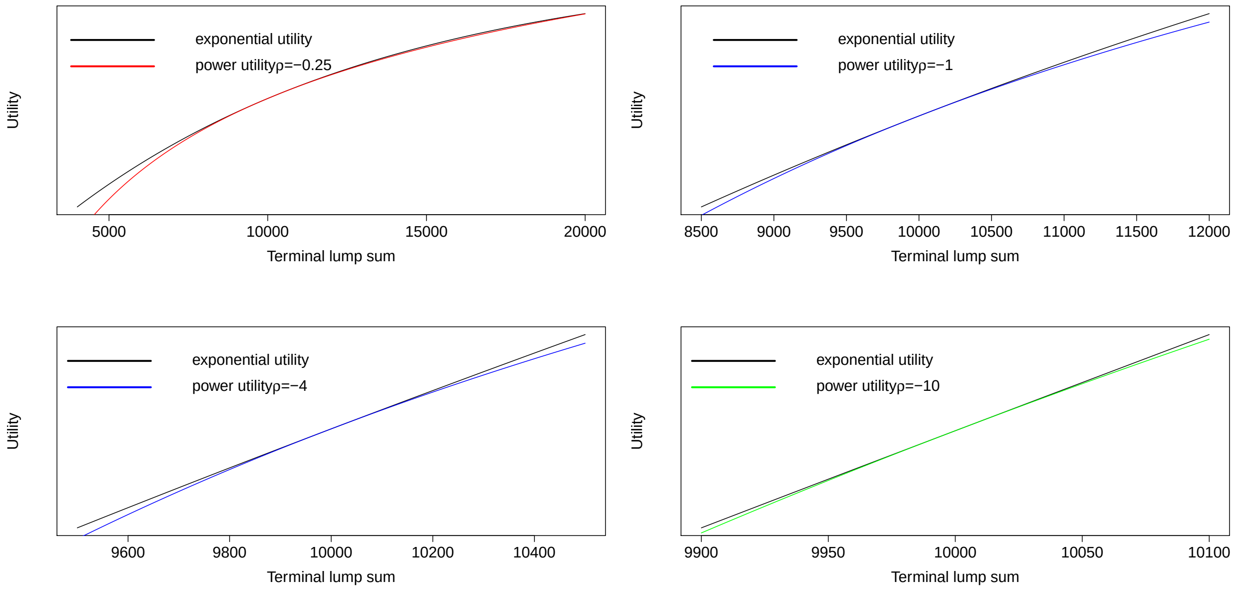

Figure 1: Comparison of the exponential utility (with parameter $\gamma = 9.77 \times 10^{-5}$ yielding $C = 9{,}480$) against the power utility with varying parameter $\rho = -0.25, -1, -4, -10$.

exponential utility. In subsequent sections, we will argue that, in our application, exponential utility is to be preferred due to its technical simplicity and intuitive investment strategy, in addition to the minimal effect of the utility choice when an upper and a lower bound restrict the range of possible outcomes. This is important when it comes to easily understandable strategies allowing for communicating the risk and being in control of the investment.

In Section 2, we introduced Lisa, John, Susan and James and assumed that they are power utility maximizers with parameters $\rho = -0.25, -1, -4, -10$, willing to invest £10,000. Figure 1 indicates that the exponential utility can approximate any power utility function locally. The more risk averse the investors, the more the area around one point must be restricted, here £10,000. Given that risk averse investors dislike significant fluctuation of their wealth from the origin, this approximation might be just good enough. This leads to the next section where we constrain the optimal strategy with an upper and lower bound for the terminal wealth. The idea is that a constrained exponential strategy can be used to approximate a power utility maximizer.

We note that the above is based on the assumption that our client has financial preferences according to an unrestricted power or exponential utility function. One might also argue that many people saving for retirement feel strongly about having a minimum outcome of their savings, knowing they can at least survive to retirement and, perhaps, keep their house and still be able to give Christmas presents to their grandchildren. Therefore, our constrained strategy can be a more natural choice. We will discuss this further in Section 6.

5. Constrained optimum with upper and lower bounds

In what follows, we consider a financial hedge using an upper and a lower bound under relative risk aversion and, for the first time through this communication, constant absolute risk aversion. To this end, the optimization of Section 4 is modified to a strategy that maximizes the terminal utility subject to the constraint $\mathbb{P}\left(G_L \leq X(T) \leq G_U\right) = 1$ or, equivalently, $\mathbb{P}\left(G_L \leq Y(T) \leq G_U\right) = 1$, meaning:

$$\begin{aligned} \max \qquad & \mathbb{E}[U(X(T))], \\ \text{subject to} \qquad & dX(t) = rX(t)\,dt + (\theta\,dt + dW(t))\,\sigma\pi(t), \quad X(0) > 0, \\ & G_L \leq X(T) \leq G_U, \quad \text{almost sure.} \end{aligned}$$

Under the assumptions that the investor obeys a power utility function and that a floor and a top apply[1], we argue later in Section 6 that the exponential utility strategy is in practice just as useful as the power utility strategy.

We define $P(t)$ as $Y^*(t)$, but with a different starting value:

$$P(t) = \begin{cases} Y^*(t)P(0)/Y(0) & \text{(power utility)} \\ Y^*(t) - Y(0) + P(0) & \text{(exponential utility)} \end{cases}.$$

We further define

$$Y^{**}(T) = \begin{cases} G_L, & \text{if } P(T) < G_L \\ P(T), & \text{if } G_L \leq P(T) \leq G_U \\ G_U, & \text{if } P(T) > G_U \end{cases}. \tag{8}$$

Our task is to show whether it is possible to find an admissible strategy which produces (8) as the final result. Indeed, the next propositions state that the portfolio is optimal and feasible in both the exponential and power utility worlds.

Proposition 1 (Constrained strategy under power utility)**.** *Define*

$$\begin{aligned} c(t, y, G_U) &= y\Phi(d_1(t, y, G_U)) - G_U e^{-r(T-t)}\Phi(d_2(t, y, G_U)), \\ p(t, y, G_L) &= G_L e^{-r(T-t)}\Phi(-d_2(t, y, G_L)) - y\Phi(-d_1(t, y, G_L)), \end{aligned}$$

where Φ is the standard normal cumulative distribution function,

$$d_{1,2}(t, y, G) = \frac{1}{\sigma A\sqrt{T-t}}\left[\ln\left(\frac{y}{G}\right) \pm \frac{1}{2}\sigma^2 A^2 (T-t)\right]$$

and A is given by (5).

Under the assumption that $G_L < Y(0) < G_U$, there exists $P(0)$ satisfying the budget con-

[1] It might be worth mentioning that, in earlier applications, authors have generally considered only downward constraints. For example, Kraft and Steffensen (2013) formalize a problem where lump sum at discrete time points is restricted by a value-at-risk or an expected shortfall constraint; Deelstra et al. (2003, 2004) consider a possibly stochastic minimum guarantee and maximize the expected utility function of the terminal wealth; Emmer et al. (2001) set up an optimization problem where they place a constraint on the capital at risk.

straint

$$Y(0) = P(0) - c(0, P(0), G_U) + p(0, P(0), G_L).$$

The solution $P(0)$ and the process

$$Y^{**}(t) = P(t) - c(t, P(t), G_U) + p(t, P(t), G_L)$$

satisfy (8) and

$$G_L \leq Y^{**}(t) \leq G_U, \text{ for all } 0 \leq t \leq T.$$

The corresponding strategy

$$\pi^{**}(t) = A[1 - \Phi(d_1(t, P(t), G_U)) - \Phi(-d_1(t, P(t), G_L))]P(t) \tag{9}$$

is optimal for the constrained problem.

Proof. See Donnelly et al. (2018). ■

Proposition 2 (Constrained strategy under exponential utility)**.** *Define*

$$\begin{aligned} c(t, y, G_U) &= R\sqrt{T-t}\, H(d(t, y, G_U)) + P(t) - G_U, \\ p(t, y, G_L) &= R\sqrt{T-t}\, H(d(t, y, G_L)), \end{aligned}$$

where R is given by (7),

$$\begin{aligned} H(x) &= x\Phi(x) + \phi(x), \\ d(t, y, G) &= \frac{G - y}{R\sqrt{T-t}}, \end{aligned} \tag{10}$$

and ϕ is the standard normal density function.

Under the assumption that $G_L < Y(0) < G_U$, there exists $P(0)$ satisfying the budget constraint

$$Y(0) = P(0) - c(0, P(0), G_U) + p(0, P(0), G_L).$$

The solution $P(0)$ and the process defined by

$$dY^{**}(t) = R(\Phi(d_U) - \Phi(d_L))(\theta dt + dW(t)),$$

with

$$d_L = d(t, P(t), G_L), \quad d_U = d(t, P(t), G_U),$$

satisfy (8). The corresponding strategy

$$\pi^{**}(t) = Ce^{-r(T-t)}(\Phi(d_U) - \Phi(d_L)), \tag{11}$$

where C is given by (6), is optimal for the constrained problem.

Proof. See Appendix A. ■

Contrary to (9), formula (11) suggests a simple financial hedging strategy of taking the initial wealth, e.g., of £10,000, of the original investment strategy and multiplying it by the probability, under the risk-neutral measure, of staying within the lower and upper bounds. This is easy to understand and will minimize the operational risk of misunderstanding and, perhaps, manually change the hedging formula in the machine room of the actuarial office of any pension provider. Most actuaries want to have an intuitive and hands-on understanding of any formula they use. They are also personally responsible if things do not work out well. It is therefore important that we have mathematical transparency as well as transparency in communication. If the product developers of the pension provider are in full control and have no doubt about the methodology they use, then this improves the chain of communication all the way through the end customer. From Propositions 1 and 2, it becomes obvious that the hedging strategy based on the exponential utility is intuitively comprehensible, contrary to the power utility, in the product development office of the pension provider. Finally, it is worth noting that the strategy holds at any future point in time; the calculation of the hedging is based on the probability of staying within the boundaries given the financial situation today.

The simplicity of the mathematical hedging of this paper makes our strategy less black-box-like and more intuitive internally in the company than, for example, strategies based on stochastic programming, which, nevertheless, can be extremely powerful and useful for pensions; see Geyer and Ziemba (2008) for a good example of dynamic stochastic programming of defined contribution schemes as considered in this paper, and also Mulvey et al. (2008) for an example in defined benefit schemes. We refer to Kraft and Munk (2011) for an intelligent life-cycle modelling of household needs that optimize consumer needs, which, nevertheless, seems to fail the simple communication requirement. It is not the purpose of the current research to provide intelligent portfolio optimization based, for example, on market timing, e.g., Luo (2017); we do believe, though, that the communicability promoted here can incorporate market timing and dynamics in the future by introducing dynamic parameters. That is beyond the scope of this work; a challenge for us remains the generalization of the current approach without compromising its transmissibility. Financial planning is of course a hot issue at the moment, but it has been for long; see Lymer and Richards (1995) and Smith and Keeney (2005) for an alternative view on financial considerations of future safety. We do have a strong belief that simplified communication is a powerful stepping-stone to optimizing financial advice. It is easier for people to match their financial planning to personal circumstances when they fully understand the financial products (see also Finke et al., 2017).

6. The value of further details in financial communication

The ease of financial communication of Section 2 implies minimization of the chance of misclassifying consumers' financial risk appetite.

In this section, we explain how direct identification of the risk aversion parameter is circumvented in our proposed communication. We start with a complex communication where the customer needs to pick the utility function parameter, the lower and the upper bound. Then, we gradually reduce to a simple communication where only the lower bound is chosen, whilst the utility parameter is given as a function of the initial wealth ($C = Y(0)$, where $C = \theta/(\sigma\gamma)$

from Equation 6) and the upper bound is consequently chosen so that it can be achieved with a 50% probability. This simple one-dimensional problem leads to the communication proposed in Section 2. We will argue that the financial gains when considering, instead, the more complex three-dimensional problem are minimal. In addition, it adds uncertainty in the communication and finding of the right parameters. Hence, in all, more complexity can easily lead to a less favourable outcome.

6.1. Three-dimensional optimization

Suppose that the financial advisor is given more freedom in selecting a detailed financial plan for the consumer based on the approach in this paper. In the first case, we let our financial advisor pick freely the lower and upper bounds as well as the underlying investment principles determined by the constant amount C invested in the risky asset. Hence, the pension saver needs to pick three parameters from a three-dimensional space of parameters. Obviously the financial communication is more complicated here as now the probabilities of reaching the upper and lower bounds must also be communicated, in addition to ensuring that the client understands the implication of selecting the underlying investment strategy and how this interacts with the hedging based on the double bounds. Extra detail in the financial communication will undoubtedly increase the likelihood of misclassification, and we know from Section 2 that this can cause significant financial losses for the client (see Table 4). In Table 5, we assume that Lisa, John, Susan and James are CRRA optimizers with utility function parameters $\rho = -0.25, -1, -4, -10$, respectively. We denote by CE_{opt} the optimal certainty equivalents under the infeasible, as ρ is practically unknown, optimal individual CRRA strategy, and by $CE_{\rho=-0.25}$ the certainty equivalents under the optimal CRRA strategy with $\rho = -0.25$, i.e., Lisa's optimal strategy. Finally, we compare those values with a constrained CARA optimal strategy. The three parameters, G_L, G_U, C, are chosen so that the certainty equivalent of the agent is maximized. We find that a constrained CARA strategy can approximate closely the optimal infeasible strategy. In particular, Lisa suffers the biggest loss and this does not exceed 2%.

Investor	Unconstrained CRRA optimization			3-D Constrained CARA optimization					
	ρ	CE_{opt}	$CE_{\rho=-0.25}$	CE	G_L	G_U	C	Prob.	% loss
Lisa	–0.25	£12,756	£12,756	£12,556	£4,050	£60,000	£9,100	0	1.56
John	–1.00	£11,643	£11,023	£11,576	£5,150	£36,300	£5,100	0	0.57
Susan	–4.00	£10,627	£6,156	£10,621	£7,000	£19,800	£1,900	0	0.05
James	–10.00	£10,280	£2,388	£10,278	£8,350	£14,600	£900	0	0.02

Table 5: Comparison of different strategies with constrained strategy derived via three-dimensional (3-D) maximization optimizing G_L, G_U and C. The four investors are assumed to obey a power utility with parameters $\rho = -0.25, -1, -4, -10$, respectively. Initial wealth is $y_0 = £10,000$ and the investment horizon is of $T = 30$ years (see additional notes in Table 3). Certainty Equivalent (CE) is the certain amount for which the investor would exchange the uncertain terminal lump sum, CE_{opt} the certainty equivalent of the infeasible optimal strategy, and $CE_{\rho=-0.25}$ the certainty equivalent under the optimal strategy for a power utility with parameter $\rho = -0.25$ (i.e., the optimal strategy for Lisa). Prob. is the probability that the terminal wealth equals the best case, G_U. % loss is the relative loss compared to the optimal strategy.

It is hard to imagine a financial advisor that would come that close to understanding Lisa's preferences from talking to her and that could estimate her power utility parameter, ρ, with an accuracy leading to a lower financial loss under a CRRA optimal strategy. CE_{opt} shows the sensitivity of the optimal strategies with respect to the right risk parameter ρ. However, also in the constrained CARA optimization, the parameters G_L, G_U, C are a priori unknown. In Table 5, it is assumed that those parameters are picked in an optimal way translating to the case of agents that perfectly understand the parameters and choose them accordingly. Obviously, this might be far-fetched as, for a true understanding, the probabilities of reaching the upper and lower bounds must be communicated as well as the nontrivial interaction with the hedging based on the double bounds. This leads us to the second case.

6.2. Two-dimensional optimization

If less communication is preferred, while still giving the financial client the opportunity to freely choose the upper and lower bounds, then one could fix the underlying unconstrained investment strategy C equal to $Y(0)$, here £10,000, but leaving the upper bound and lower bound to be decided by the client. This way the pension saver needs to pick two parameters from a two-dimensional space of parameters. This is less complicated than the previous detailed communication, but still more than the strategy of Section 2. The financial losses compared to the infeasible power utility strategy are given in Table 6, with the highest financial loss being just 2.7%.

Investor	Unconstrained CRRA optimization			2-D Constrained CARA optimization					
	ρ	CE_{opt}	$CE_{\rho=-0.25}$	CE	G_L	G_U	C	Prob.	% loss
Lisa	–0.25	£12,756	£12,756	£12,550	£4,300	£60,000	£10,000	0.00	1.61
John	–1.00	£11643	£11,023	£11,324	£6,900	£20,200	£10,000	0.21	2.74
Susan	–4.00	£10,627	£6,156	£10,430	£8,850	£11,800	£10,000	0.63	1.85
James	–10.00	£10,280	£2,388	£10,184	£9,500	£10,700	£10,000	0.69	0.93

Table 6: Comparison of different strategies with constrained strategy derived via two-dimensional (2-D) maximization optimizing G_L, G_U. C is fixed at the initial wealth. The four investors are assumed to obey a power utility with parameters $\rho = -0.25, -1, -4, -10$, respectively. Initial wealth is $y_0 =$ £10,000 and the investment horizon is of $T = 30$ years (see additional notes in Table 5).

Again this loss is so small that it is hard to imagine a financial advisor that could estimate the infeasible constrained power utility parameter better than that.

6.3. Easy communication: The one-dimensional case

In the final case, we simplify the communication further to the setting of Section 2: see Table 7. The highest financial loss compared to the infeasible strategy is 5.8%, but even this is so low that we cannot imagine a real-life financial advisor approaching Lisa's unknown financial preferences more than our proposed financial strategy.

6.4. Additional comments

We note that the previous discussion is based on the assumption that our client has financial preferences according to an unrestricted power utility function. One could also argue that many people saving for retirement feel strongly about having a minimum outcome of their savings so that they can at least survive to retirement, i.e., for instance, be able to keep their house and pay their rent. If that is the case, then a reversed comparison where an unconstrained

Investor	Unconstrained CRRA optimization			1-D Constrained CARA optimization					
	ρ	CE_{opt}	$CE_{\rho=-0.25}$	CE	G_L	G_U	C	Prob.	% loss
Lisa	–0.25	£12,756	£12,756	£12,017	£3,650	£16,500	£10,000	0.5	5.79
John	–1.00	£11,643	£11,023	£11,264	£6,500	£15,200	£10,000	0.5	3.26
Susan	–4.00	£10,627	£6,156	£10,416	£9,100	£12,300	£10,000	0.5	1.99
James	–10.00	£10,280	£2,388	£10,171	£9,650	£11,000	£10,000	0.5	1.06

Table 7: Comparison of different strategies with constrained strategy derived via one-dimensional (1-D) maximization optimizing G_L. The upper bound G_U is uniquely defined as the value which can be reached with a probability of 50% and C is fixed at the initial wealth. The four investors are assumed to obey a power utility with parameters $\rho = -0.25, -1, -4, -10$, respectively. Initial wealth is $y_0 = \pounds 10{,}000$ and the investment horizon is of $T = 30$ years (see additional notes in Table 5).

CRRA strategy is wrongly followed – independent of the chosen risk aversion parameter – would lead to a utility of minus infinity. This means that, not only is the power utility function hard to communicate and estimate in praxis, but it might also at the same time not represent many pension savers' risk preferences. We further argue that the majority of pension savers entrusting their money to a pension provider would oppose gambling. If we define gambling as having a financially beneficial outcome in a minority of cases at the cost of a significantly worse outcome in most cases, then that would lead us to exactly the risk preferences suggested in our communication. If the risk preferences suggested by our financial strategy are exactly matching the client's, then our approach is both financially optimal in a mathematical sense and easy to communicate. We believe this to be the case.

7. Extension to annuities

Useful as our development of the lump sum case is, it unblocks the road to universality. In particular, we mention here annuities involving a random number of random future monthly payments. When adapting to annuities, the product of Section 2 offered to the customer is altered as follows: the worst case scenario is that the annuity will be reduced after a certain number of years of guaranteed high payments; the best case scenario is that the annuity will continue at the same high level life-long.

In a recent research event titled "Self-selection and Risk Sharing in a Modern World of Life-Long Annuities" hosted by the Actuarial Research Centre of the Institute and Faculty of Actuaries, practitioners and academics were invited to discuss a presentation based on these ideas; we refer to Gerrard et al. (2018) for a detailed treatment of this extension. More specifically, building upon the current approach in this paper, the authors, first, introduce a mortality pooling approach defined in Bräutigam et al. (2017) to ensure that a pool of individuals can hedge mortality risk without involving intermediaries. Second, they explain how investment risk pooling and hedging of an inflation fund can ensure that payments are communicated and guaranteed in real terms. Finally, the worst and best cases are reformulated as aforesaid.

8. Conclusion

In this paper, we suggest a combination of constant absolute risk aversion with an upper and a lower bound to meet Merton's vision of creating a transparent platform for pension products. We focus on real returns and introduce an inflation fund without risk as well as a risky fund based on stocks. A single investment strategy with a lump-sum payment is then considered; this is part of most pension schemes, so it is important itself. This way we learn how to communicate risk and allow the pension saver self-select his simple pension decision case, before considering more complicated actuarial constructions like life annuity products. We conclude that self-selection is possible via our intuitive and easy-to-understand combination of constant absolute risk aversion and double bounds. Future research is targeted towards generalizing the results in a situation with risk in the inflation fund as well as more complicated actuarial pension constructions, including mortality-dependent annuity products, and financial investment constructions including market timing.

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Appendix A. Proof of Proposition 2

We define the martingale measure $\mathbb{Q}$ such that $W^{\mathbb{Q}}(t) = W(t) + \theta t$ is a standard Brownian motion. With this in mind, we write

$$P(t) = P(0) + RW^{\mathbb{Q}}(t).$$

Conditional on the history of the process until time $t > 0$,

$$P(T) = P(0) + R(W^{\mathbb{Q}}(t) + \sqrt{T-t}\,Z),$$

where Z is a standard normal random variable under $\mathbb{Q}$. We note that

$$P(T) > G_U \iff W^{\mathbb{Q}}(t) + \sqrt{T-t}\,Z > R^{-1}\left(G_U - P(0)\right) \iff Z > d_U,$$

where

$$d_U = \frac{1}{\sqrt{T-t}}\left[R^{-1}\left(G_U - P(0)\right) - W^{\mathbb{Q}}(t)\right]$$

and, similarly, we have that $P(T) < G_L$ is the same as the event $Z < d_L$, where

$$d_L = \frac{1}{\sqrt{T-t}}\left[R^{-1}\left(G_L - P(0)\right) - W^{\mathbb{Q}}(t)\right].$$

$Y(t)$ is given by the present value of the portfolio at time t under $\mathbb{Q}$:

$$\begin{aligned}
Y(t) &= \mathbb{E}_{\mathbb{Q}}\left(\max\left(G_L, \min\left(G_U, P(T)\right)\right) \middle| \mathcal{F}_t^{\mathbb{Q}} \right) \\
&= \mathbb{E}_{\mathbb{Q}}\left(P(T) \middle| \mathcal{F}_t^{\mathbb{Q}} \right) + \mathbb{E}_{\mathbb{Q}}\left(\max(G_L - P(T), 0) \middle| \mathcal{F}_t^{\mathbb{Q}} \right) - \mathbb{E}_{\mathbb{Q}}\left(\max(P(T) - G_U, 0) \middle| \mathcal{F}_t^{\mathbb{Q}} \right) \\
&= \int_{-\infty}^{d_L} G_L \phi(z)\, dz + \int_{d_U}^{\infty} G_U \phi(z)\, dz + \int_{d_L}^{d_U} \left(P(0) + R(W^{\mathbb{Q}}(t) + \sqrt{T-t}z) \right) \phi(z)\, dz \\
&= G_L \Phi(d_L) + G_U \left[1 - \Phi(d_U)\right] + \left(P(0) + R W^{\mathbb{Q}}(t) \right) \left[\Phi(d_U) - \Phi(d_L)\right] \\
&\quad - R\sqrt{T-t}\left[\phi(d_U) - \phi(d_L)\right] \\
&= G_U - R\sqrt{T-t}\left[H(d_U) - H(d_L)\right],
\end{aligned}$$

where H is given by (10). As $H'(x) = \Phi(x) \in (0,1)$ and $d_L < d_U$, we deduce that

$$0 \leq H(d_U) - H(d_L) \leq d_U - d_L = \frac{1}{\sqrt{T-t}} R^{-1}(G_U - G_L),$$

confirming that $\mathbb{P}\left(G_L \leq Y(t) \leq G_U\right) = 1$ for all t.

Returning to the standard measure $\mathbb{P}$, we can write both d_L and d_U as functions of t and $w = W(t)$:

$$\begin{aligned}
d_L(t,w) &= \frac{1}{\sqrt{T-t}} \left[R^{-1}\left(G_L - P(0)\right) - w - \theta t \right], \\
d_U(t,w) &= \frac{1}{\sqrt{T-t}} \left[R^{-1}\left(G_U - P(0)\right) - w - \theta t \right],
\end{aligned}$$

with

$$\frac{\partial d_L}{\partial t} = -\frac{\theta}{\sqrt{T-t}} + \frac{d_L}{2(T-t)}, \qquad \frac{\partial d_L}{\partial w} = -\frac{1}{\sqrt{T-t}},$$

and similarly for d_U. By exploiting the expressions for d_L and d_U, we rewrite $Y(t) = \eta(t, W(t))$, where η satisfies

$$\begin{aligned}
\frac{\partial \eta}{\partial t} &= \frac{R}{2\sqrt{T-t}}\left[H(d_U) - H(d_L)\right] - R\sqrt{T-t}\left[H'(d_U)\frac{\partial d_U}{\partial t} - H'(d_L)\frac{\partial d_L}{\partial t} \right] \\
&= \frac{R}{2\sqrt{T-t}}\left[H(d_U) - H(d_L)\right] \\
&\quad + R\theta\left[\Phi(d_U) - \Phi(d_L)\right] - \frac{R}{2\sqrt{T-t}}\left[d_U \Phi(d_U) - d_L \Phi(d_L)\right] \\
&= \frac{R}{2\sqrt{T-t}}\left[\phi(d_U) - \phi(d_L)\right] + R\theta\left[\Phi(d_U) - \Phi(d_L)\right], \\
\frac{\partial \eta}{\partial w} &= -R\sqrt{T-t}\left(H'(d_U)\frac{\partial d_U}{\partial w} - H'(d_L)\frac{\partial d_L}{\partial w} \right) = R\left[\Phi(d_U) - \Phi(d_L)\right], \\
\frac{\partial^2 \eta}{\partial w^2} &= -\frac{R}{\sqrt{T-t}}\left[\phi(d_U) - \phi(d_L)\right],
\end{aligned}$$

so that

$$dY(t) = \left(\frac{\partial \eta}{\partial t} + \frac{1}{2}\frac{\partial^2 \eta}{\partial w^2} \right) dt + \frac{\partial \eta}{\partial w}\, dW(t) = R\left(\Phi(d_U) - \Phi(d_L)\right)\left(\theta\, dt + dW(t)\right).$$

From (1),

$$dY(t) = \sigma e^{r(T-t)} \pi(t) \left(\theta \, dt + dW(t)\right),$$

suggesting that the strategy is given by

$$\pi^{**}(t, y) = C e^{-r(T-t)} \left(\Phi(d_U) - \Phi(d_L)\right),$$

giving rise to the terminal expression (8) for $Y^{**}(T)$.

Let $V_0(t, y)$ be the value function of the proposed solution:

$$V_0(t, y) = \mathbb{E}\left(-\frac{1}{\gamma} e^{-\gamma Y(T)} | Y(t) = y\right).$$

We demonstrate the optimality of π^{**} by showing that V_0 satisfies the Hamilton–Jacobi–Bellman equation and that π^{**} is the strategy giving rise to $Y(t)$. We are faced with the problem that $Y(t)$ is only defined as a function of $W(t)$ and t. We therefore write

$$V_0\left(t, Y(t)\right) = V_0\left(t, \eta\left(t, W(t)\right)\right) = \bar{V}\left(t, W(t)\right),$$

so that

$$\begin{aligned}
\frac{\partial \bar{V}}{\partial t} &= \frac{\partial V_0}{\partial t} + \frac{\partial V_0}{\partial y}\frac{\partial \eta}{\partial t}, && \text{(A.1)}\\
\frac{\partial \bar{V}}{\partial w} &= \frac{\partial V_0}{\partial y}\frac{\partial \eta}{\partial w}, \quad \frac{\partial^2 \bar{V}}{\partial w^2} = \frac{\partial^2 V_0}{\partial y^2}\left(\frac{\partial \eta}{\partial w}\right)^2 + \frac{\partial V_0}{\partial y}\frac{\partial^2 \eta}{\partial w^2}. && \text{(A.2)}
\end{aligned}$$

Now

$$P(T) = P(0) + R\left(\theta T + W(T)\right) \stackrel{\mathcal{D}}{=} P(0) + R\left(\theta T + W(t) + \sqrt{T-t} Z\right),$$

where Z is a standard normal random variable under the original probability measure $\mathbb{P}$. As a result,

$$P(T) > G_U \iff Z > D_U(t, w) \stackrel{def}{=} d_U(t, w) - \theta\sqrt{T-t}$$

(D_L follows similarly from $P(T) < G_L$). Given the previous definition, we get

$$\begin{aligned}
\bar{V}\left(t, w\right) &= \mathbb{E}\left(-\frac{1}{\gamma} e^{-\gamma Y(T)} | W(t) = w\right)\\
&= -\frac{1}{\gamma}\left(\int_{-\infty}^{D_L} e^{-\gamma G_L}\phi(z)\, dz + \int_{D_U}^{\infty} e^{-\gamma G_U}\phi(z)\, dz + \int_{D_L}^{D_U} e^{-\gamma\left(P(0) + R(\theta T + w + \sqrt{T-t} z)\right)}\phi(z)\, dz\right)\\
&= -\frac{1}{\gamma}\left(e^{-\gamma G_L}\Phi(D_L) + e^{-\gamma G_U}\left(1 - \Phi(D_U)\right)\right.\\
&\quad \left. + e^{-\gamma P(0) - \frac{1}{2}\theta^2(T+t) - \theta w}\left(\Phi(D_U + \theta\sqrt{T-t}) - \Phi(D_L + \theta\sqrt{T-t})\right)\right),
\end{aligned}$$

with

$$\begin{aligned}
\frac{\partial \bar{V}}{\partial w} &= -\frac{1}{\gamma}\left(e^{-\gamma G_U}\frac{\phi(D_U)}{\sqrt{T-t}} - e^{-\gamma G_L}\frac{\phi(D_L)}{\sqrt{T-t}} - \theta e^{-\gamma P(0) - \frac{1}{2}\theta^2(T+t) - \theta w}\left(\Phi(d_U) - \Phi(d_L)\right)\right.\\
&\quad \left. - e^{-\gamma P(0) - \frac{1}{2}\theta^2(T+t) - \theta w}\left(\frac{\phi(d_U)}{\sqrt{T-t}} - \frac{\phi(d_L)}{\sqrt{T-t}}\right)\right).
\end{aligned}$$

As

$$\begin{aligned}
\phi(D_U) = \frac{e^{-\frac{1}{2}d_U^2+\theta d_U\sqrt{T-t}-\frac{1}{2}\theta^2(T-t)}}{\sqrt{2\pi}} &= \phi(d_U)e^{-\frac{1}{2}\theta^2(T-t)+\gamma G_U-\gamma P(0)-\theta w-\theta^2 t} \\
&= \phi(d_U)e^{\gamma G_U-\gamma P(0)-\frac{1}{2}\theta^2(T+t)-\theta w}
\end{aligned}$$

(similarly for D_L), we get that

$$\frac{\partial \bar{V}}{\partial w} = Re^{-\gamma P(0)-\frac{1}{2}\theta^2(T+t)-\theta w}\left(\Phi(d_U)-\Phi(d_L)\right)$$

and, consequently,

$$\frac{\partial^2 \bar{V}}{\partial w^2} = Re^{-\gamma P(0)-\frac{1}{2}\theta^2(T+t)-\theta w}\left(-\theta\left(\Phi(d_U)-\Phi(d_L)\right)-\frac{1}{\sqrt{T-t}}\left(\phi(d_U)-\phi(d_L)\right)\right).$$

Then, from (A.2),

$$\begin{aligned}
\frac{\partial V_0}{\partial y} &= e^{-\gamma P(0)-\frac{1}{2}\theta^2(T+t)-\theta w}, \\
\frac{\partial^2 V_0}{\partial y^2} &= \frac{\frac{\partial^2 \bar{V}}{\partial w^2}-\frac{\partial V_0}{\partial y}\frac{\partial^2 \eta}{\partial w^2}}{\left(\frac{\partial \eta}{\partial w}\right)^2} = -\gamma\frac{e^{-\gamma P(0)-\frac{1}{2}\theta^2(T+t)-\theta w}}{\Phi(d_U)-\Phi(d_L)},
\end{aligned}$$

and from (A.1),

$$\begin{aligned}
\frac{\partial V_0}{\partial t} &= e^{-\gamma P(0)-\frac{1}{2}\theta^2(T+t)-\theta w}\left(\frac{R\theta}{2}\left(\Phi(d_U)-\Phi(d_L)\right)+\frac{R}{2\sqrt{T-t}}\left(\phi(d_U)-\phi(d_L)\right)\right) \\
&\quad -e^{-\gamma P(0)-\frac{1}{2}\theta^2(T+t)-\theta w}\left(R\theta\left(\Phi(d_U)-\Phi(d_L)\right)+\frac{R}{2\sqrt{T-t}}\left(\phi(d_U)-\phi(d_L)\right)\right) \\
&= -\frac{R\theta}{2}e^{-\gamma P(0)-\frac{1}{2}\theta^2(T+t)-\theta w}\left(\Phi(d_U)-\Phi(d_L)\right),
\end{aligned}$$

from which

$$\frac{\partial V_0}{\partial t}-\frac{\theta^2}{2}\frac{\left(\frac{\partial V_0}{\partial y}\right)^2}{\frac{\partial^2 V_0}{\partial y^2}} = 0$$

follows.

Finally, we prove that it is possible to choose $P(0)$ in such a way that the budget constraint $X(0)=x_0$ is satisfied. The budget constraint is

$$\begin{aligned}
e^{rT}x_0 = Y(0) &= \eta(0,0) \\
&= G_U - R\sqrt{T}\left[H\left(R^{-1}\frac{G_U-P(0)}{\sqrt{T}}\right)-H\left(R^{-1}\frac{G_L-P(0)}{\sqrt{T}}\right)\right]
\end{aligned}$$

with its derivative with respect to $P(0)$ given by

$$\Phi\left(R^{-1}\frac{G_U-P(0)}{\sqrt{T}}\right)-\Phi\left(R^{-1}\frac{G_L-P(0)}{\sqrt{T}}\right) > 0.$$

The smallest and largest possible values are therefore the limits as $P(0) \to \pm\infty$: at the top end,

$$G_U - R\sqrt{T} \lim_{q\to\infty} \int_{R^{-1}(G_L-q)/\sqrt{T}}^{R^{-1}(G_U-q)/\sqrt{T}} \Phi(z)\, dz = G_U,$$

and at the bottom end,

$$\begin{aligned} & G_U - R\sqrt{T} \lim_{q\to-\infty} \int_{R^{-1}(G_L-q)/\sqrt{T}}^{R^{-1}(G_U-q)/\sqrt{T}} \Phi(z)\, dz \\ = \quad & G_U - R\sqrt{T} \left(\frac{R^{-1}(G_U-q)}{\sqrt{T}} - \frac{R^{-1}(G_L-q)}{\sqrt{T}} \right) = G_L, \end{aligned}$$

as expected. We conclude that it is always possible to find a value of $P(0)$ such that the budget constraint is satisfied as long as

$$G_L < e^{rT} x_0 < G_U.$$

Assuming that this inequality holds, we have a strategy which is feasible and has a value function satisfying the Hamilton–Jacobi–Bellman equation: we conclude that this must be the optimal strategy.

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