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Actuarial Research Centre (ARC)

PhD studentship output

Optimal Design of Structured Products

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Outline

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Introduction

- Our aim is to design new structured products (Pension) that can be used by insurance company to smooth investment return for their customers. The focus is on new mechanisms for investment risk sharing between customers and the insurance company.

Introduction

- What is optimal?
  largest expected terminal wealth or largest expected utility of terminal wealth?
  - St. Petersburg Paradox
    A game is tossing a fair coin until the first head appears. What you will receive is \$ 2^n if the first head appears at n^{th} coin tossing. How much would you pay to play this game?

$$E = \frac{1}{2} \times 2 + \frac{1}{2^2} \times 2^2 + ... + \frac{1}{2^n} \times 2^n + ... = \infty \qquad (1)$$

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Expected Utility Theory

The expected utility of an uncertain prospect $(x_1, p_1; x_2, p_2; ...x_n, p_n)$ is the utility of each outcome weighted by its probability.

$$E[U(x_1, p_1; x_2, p_2; ...x_n, p_n)] = p_1 U(x_1) + p_2 U(x_2) + ... + p_n U(x_n) \quad (2)$$

- Characteristics

1. Non satiation.

People always prefer more to less. If $x_1 > x_2$ then $U(x_1) > U(x_2)$.

$$U'(x) > 0 \quad (3)$$

2. Risk aversion.

People tend to favor an investment with certain payoff rather than another investment with same but uncertain expected payoff.

$$U(\frac{x_1 + x_2}{2}) > \frac{1}{2}U(x_1) + \frac{1}{2}U(x_2) \rightarrow U''(x) < 0 \quad (4)$$

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Expected Utility Theory

- Some popular forms of utility functions are:

$$U(X) = 1 - \exp(-aX), \quad a > 0 \tag{5}$$

$$U(X) = \ln(X) \tag{6}$$

$$U(X) = \frac{X^\gamma}{\gamma}, \quad \gamma < 1, \gamma \neq 0 \tag{7}$$

- Limitations: Reference point, Non-linear probability, risk seeking, loss aversion.

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Cumulative Prospect Theory

Tversky and Kahneman (1992) proposed the Cumulative Prospect Theory (CPT) which better explains people's behaviour in decision making under uncertainty. The main distinction of CPT (and its predecessor Prospect Theory) is that people tend to think of possible outcomes relative to a certain reference point (often the status quo) rather than to the final status. The carriers of value are gains and loss, rather than final wealth. Additionally, risk aversion and risk seeking are determined jointly by the value function and weighting function rather than by value function only in EUT.

Cumulative Prospect Theory

- Model:

Let f denote the prospect $(x_{-m}, p_{-m}; x_{-m+1}, p_{-m+1}; \ldots; x_0, p_0; \ldots; x_{n-1}, p_{n-1}; x_n, p_n;)$ where x_i is the outcome and p_i is the corresponding probability. In CPT, outcomes are arranged in ascending order. Positive subscript denotes positive outcome, negative subscript denotes negative outcome and zero subscript denotes neutral outcome. CPT values gains and loss separately. The utility of a prospect is the sum of utility of positive prospect f^+ and negative prospect f^-. The formula is given as:

$$V(f) = V(f^+) + V(f^-) = \sum_{i=0}^{n} \pi_i^+ v(x_i) + \sum_{i=-m}^{0} \pi_i^- v(x_i) \qquad (8)$$

where v is the value function and π is the weighting function.

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Cumulative Prospect Theory

The value function is monotonically increasing. Positive outcome has positive value while negative outcome has negative value. The value function is concave in the positive part and convex in the negative part.

$$v(x) = \begin{cases} x^\alpha \text{ if } x \geq 0 \\ -\lambda(-x)^\beta \text{ if } x < 0 \end{cases} \quad (9)$$

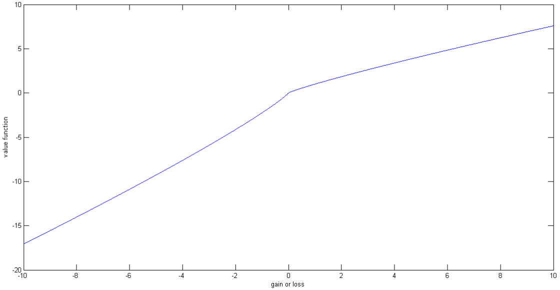

Figure: Value function of CPT, $\lambda = 2.25$ and $\alpha = \beta = 0.88$.

The weighting function π is defined by capacity function w which is a transformation of probabilities:

$$\pi_n^+ = w^+(p_n), \quad \pi_{-m}^- = w^-(p_{-m}) \tag{10}$$

$$\pi_i^+ = w^+(pi + ... + p_n) - w^+(p_{i+1} + ... + p_n), 0 \leq i \leq n-1 \tag{11}$$

$$\pi_i^- = w^-(p_{-m} + ... + p_i) - w^-(p_{-m} + ... + p_{i-1}), 1-m \leq i \leq 0. \tag{12}$$

$$w^+(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{\frac{1}{\gamma}}} \tag{13}$$

$$w^-(p) = \frac{p^\delta}{(p^\delta + (1-p)^\delta)^{\frac{1}{\delta}}} \tag{14}$$

The capacity w satisfies $w(\phi) = 0$ and $w(S) = 1$ where ϕ is the empty set and S is the full set.

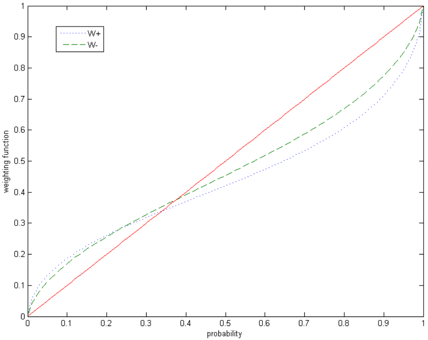

Figure: Weighting function of CPT, $\gamma = 0.61$ and $\delta = 0.69$.

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Cumulative Prospect Theory

- Characteristics

The shape of the value function is an "S". The value function is concave in positive part and convex in negative part. The curvature for losses is steeper than for gains.
The shape of weighting function looks like an inverse "S". Over-valuing small probabilities shows risk seeking for gains and risk aversion for losses of small probabilities.These results explain the existence of lottery and insurance. Under-valuing of moderate and high probabilities gives rise to risk aversion in choices between sure gains and probable gains, and risk seeking in choices between sure losses and probable losses. The effects are also enhanced by the curvature of the value function.

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New Contract

Guillén, Jørgensen and Nielsen (2006) (GJN's contract) introduced a popular pension product in Denmark.

- Structure

Customer pays a one-off premium P at start and receives his principal and investment return when the contract expires. The maturity of the contract is T years. The premium is invested in an equity index. Each year, customers' individual account D increases at a guaranteed rate g. Apart from this guaranteed return, Customers also can receive a proportion from smoothing account.
The equity index is assumed increases at a rate of R_n in year n. So the value of equity index at the end of year n is:

$$A_n = \begin{cases} P, & n = 0 \\ (1 + R_n)A_{n-1}, & n \in \{1, 2, 3, ..., N\}. \end{cases} \tag{15}$$

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New Contract

Then the value of customer account at the end of year n can be presented as :

$$D_n = \begin{cases} P, & n = 0 \\ (1+g)D_{n-1} + \alpha(A_n - (1+g)D_{n-1}), & n \in \{1, ..., N\}. \end{cases} \quad (16)$$

Recall that people prefer insurance (capital protection) and lottery (a participation in the gains), we modify the above contract and propose a new contract. In this new contract, customer receives a non-negative proportion of the smoothing account (bonus). The evolution of the value of customer account is given as

$$D_n = \begin{cases} P, & n = 0 \\ (1+g)D_{n-1} + \alpha \max((A_n - (1+g)D_{n-1}), 0), & n \in \{1, ..., N\}. \end{cases} \quad (17)$$

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New Contract

- Pricing

Let C_n denotes the payoff of nth year option, that is

$$C_n = \max((A_n - (1+g)D_{n-1}), 0). \quad (18)$$

After recursive substitution of non-trivial part of equation (17), we get

$$D_N = D_0(1+g)^N + \alpha \sum_{i=1}^{N} C_i(1+g)^{N-i} \quad (19)$$

For Simplicity, risk free rate r_f is assumed constant. Then

$$\begin{aligned} V(0, D_N) &= \tfrac{1}{(1+r_f)^N} E^Q\{D_N\} \\ &= (\tfrac{1+g}{1+r_f})^N D_0 + \tfrac{\alpha}{(1+r_f)^N} E^Q\{\textstyle\sum_{i=1}^N [A_i - (1+g)D_{i-1}]_+ (1+g)^{N-i}\} \end{aligned} \quad (20)$$

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New Contract

As this model is path dependent, no analytic formula can be given. Monte Carlo method is used to price this contract. The equity index price is assumed follows Black-Scholes model. The drift μ and the volatility σ of the underlying equity are also assumed to be constant. The W_t is a standard Brownian motion. The dynamics of the underlying equity is

$$dA_t = \mu A_t dt + \sigma A_t dW_t, \tag{21}$$

$$A_0 = P. \tag{22}$$

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New Contract

The contract is viewed as a fair contract if $V(0, D_N) = P$ under risk neutral measure. The relationship between guarantee rate g and participation rate α is presented in the following figure.

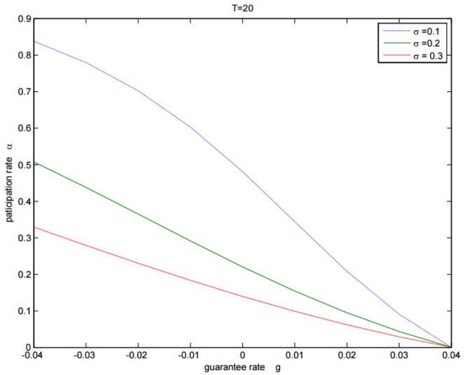

Figure: $T = 20$ years, $r_f = 0.04$ and $\sigma = 0.1, 0.2, 0.3$.

New Contract

- Result:

In order to examine if our new contract outperforms GJN's contract, 100,000 equity index price paths are simulated under real world measure $\mathbb{P}$. The mean return, standard deviation of return and utility of each asset in the scenario ($\mu = 0.065$, $\sigma = 0.15$) are given in the following table.

	GJN	New Contract	Bond	Equity Index
Mean return	3.0612	2.8793	2.1911	3.5243
Std of return	1.6829	1.4889	0	2.6340
Expected CPT Utility	1.2816	1.3330	1.0914	1.2785

Table: $\alpha = 0.1286$. $T = 20$ years, $r_f = 0.04$, $\mu = 0.065$, $g = 0.02$ and $\sigma = 0.15$, $P = 1$.

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New Contract

We also calculate the proportion of each asset in the optimal portfolio under CPT.

	GJN	New Contract	Bond	Equity Index
weight	0	0.61	0	0.39

Table: $\alpha = 0.1286$. $T = 20$ years, $r_f = 0.04$, $\mu = 0.065$, $g = 0.02$ and $\sigma = 0.15, P = 1$.

CPT utility of this optimized portfolio is 1.3747 which is larger than new model and underlying asset.

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New Contract

As customers tend to choose pension product with the term best match their pension plan. Some figures showing the changes of the proportion in optimal portfolio with different terms are plotted.

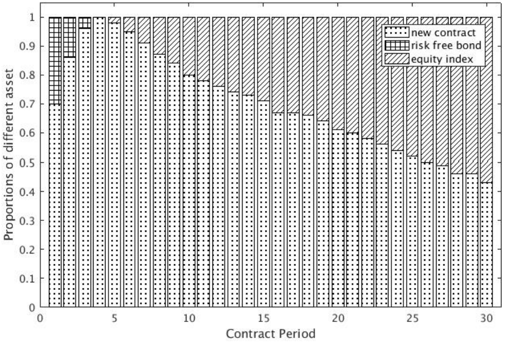

Figure: The composition of optimal portfolio for different terms. $r_f = 0.04$, $g = 0.02$, $\mu = 0.065$ and $\sigma = 0.15$.

New Contract

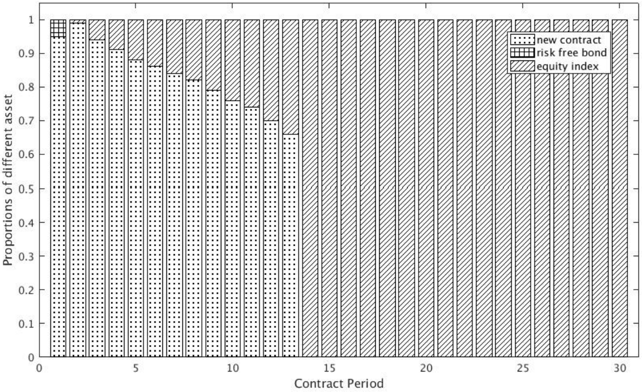

Figure: The composition of optimal portfolio for different terms. $r_f = 0.04$, $g = 0.02$, $\mu = 0.1$ and $\sigma = 0.2$.

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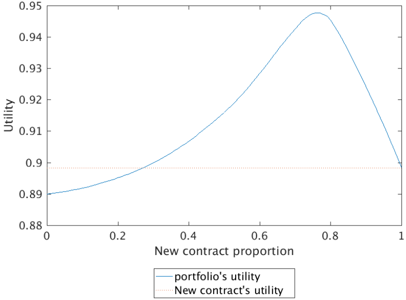

Figure: The composition of optimal portfolio for term of $T = 10$. $r_f = 0.04$, $g = 0.02$, $\mu = 0.1$, $\sigma = 0.2$.

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New Contract

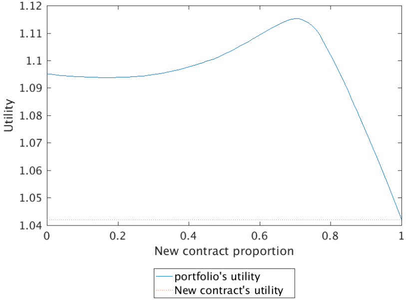

Figure: The composition of optimal portfolio for term of $T = 12$. $r_f = 0.04$, $g = 0.02$, $\mu = 0.1$, $\sigma = 0.2$.

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New Contract

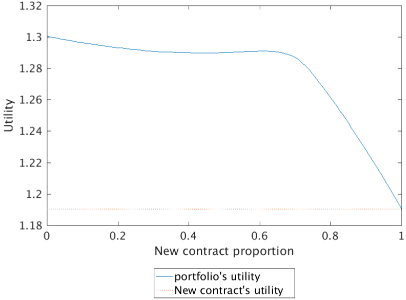

Figure: The composition of optimal portfolio for term of $T = 14$. $r_f = 0.04$, $g = 0.02$, $\mu = 0.1$, $\sigma = 0.2$.

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New Contract

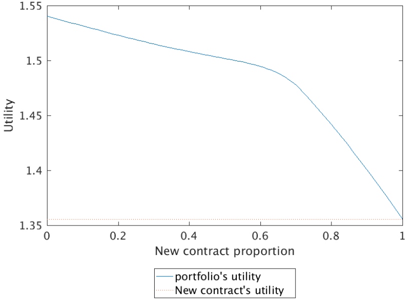

Figure: The composition of optimal portfolio for term of $T = 16$. $r_f = 0.04$, $g = 0.02$, $\mu = 0.1$, $\sigma = 0.2$.

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New Contract

- Conclusion: In this paper, we examined a new pension contract with characteristics of guarantees and bonuses using cumulative prospect theory. Under cumulative prospect theory, the contract gives higher utility than the contract introduced in Guillén, Jørgensen and Nielsen (2006). With the increase of policyholder's investment period, the proportion of risky asset in optimal portfolio also increases while the proportion of risk free asset decreases. This result conforms traditional pension investment advice.