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Optimal Design of Pension Products

Zhaoxun Mei Catherine Donnelly Torsten Kleinow

Heriot Watt University, Actuarial Research Centre

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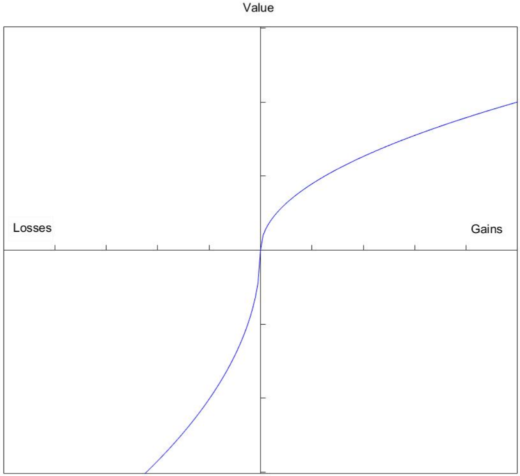
Motivation

- What kind of pension contract is optimal to customers?

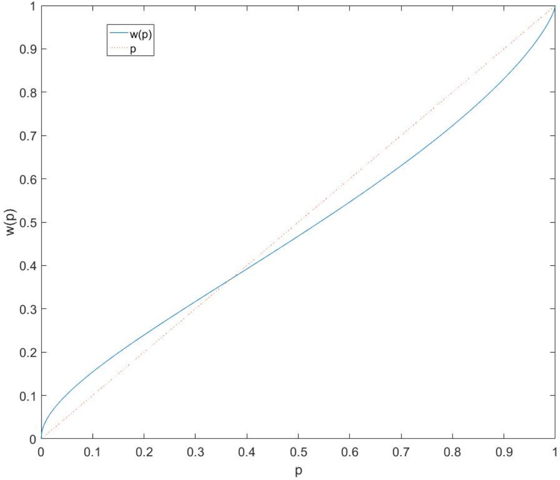
Cumulative Prospect Theory

- Expected utility theory
  Limitations: framing effects, non-linear preference (Allais's paradox), source dependence (Ellsberg's paradox), risk seeking, loss aversion.
- Cumulative Prospect theorz In order to explain the above violations, Tversky and Kahneman (1992) proposed the Cumulative Prospect Theory (CPT). It is viewed as a better model in explaining people's behaviour in decision making under uncertainty.

Value Function



Weighting function



Cumulative Prospect Theory

- Model:

Let f denote a prospect $(x_{-m}, p_{-m}; x_{-m+1}, p_{-m+1}; ...; x_0, p_0; ...; x_{n-1}, p_{n-1}; x_n, p_n;)$ where x_i is the outcome and p_i is the corresponding probability. In CPT, outcomes are arranged in ascending order. CPT values gains and loss separately. The utility of a prospect is the sum of utility of positive prospect f^+ and negative prospect f^-. The formula is given as:

$$V(f) = V(f^+) + V(f^-) = \sum_{i=0}^{n} \pi_i^+ v(x_i) + \sum_{i=-m}^{0} \pi_i^- v(x_i) \quad (1)$$

where v is the value function and π_i is the decision weights associated with outcome x_i.

Cumulative Prospect Theory

The value function is monotonically increasing. Positive outcome has positive value while negative outcome has negative value. The value function is concave in the positive part and convex in the negative part. The curvature for losses is steeper than for gains.

$$v(x) = \begin{cases} x^{\alpha} \text{ if } x \geq 0 \\ -\lambda(-x)^{\beta} \text{ if } x < 0 \end{cases} \tag{2}$$

Cumulative Prospect Theory

The decision weight π is defined by weighting function w.

$$\pi_i^+ = w(pi + \ldots + p_n) - w(p_{i+1} + \ldots + p_n), 0 \leq i \leq n-1; \tag{3}$$
$$\pi_i^- = w(p_{-m} + \ldots + p_i) - (p_{-m} + \ldots + p_{i-1}), 1 - m \leq i \leq 0; \tag{4}$$
$$\pi_n^+ = w(p_n); \tag{5}$$
$$\pi_{-m}^- = w(p_{-m}). \tag{6}$$

The weighting function w rescales the probabilities and satisfies $w(0) = 0$ and $w(1) = 1$.

$$w(p) = e^{-(-\ln p)^{\varphi}}. \tag{7}$$

New Contract

- Financial market model

The evolution of the value of the risk free asset (bond) is given as:

$$dB_t = rB_t dt, \qquad B_0 = b, \tag{8}$$

where r is the constant risk free interest rate. The price of the risky asset (equity index) are assumed follows the geometric Brownion motion.

$$\begin{cases} S_0 &= s \\ dS_t &= \mu S_t dt + \sigma S_t dW_t. \end{cases} \tag{9}$$

where the expected growth rate μ , and volatility σ are positive constants. W_t is a Wiener process.

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New Contract

- Structure

The customer is assumed to pay a one-off premium $P > 0$ at start. The premium is credited into the **investment account** A which is a notional account mimicking the trend of the risky asset. Then the value of the investment account is:

$$A_t = P \cdot \exp[(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t], \quad t \in [0, T] \tag{10}$$

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New Contract

What the customer receive at the expiration day is the terminal value in the **customer account** D. The value of customer account in new contract at the end of year n can be presented as :

$$D_n = \begin{cases} P, & n = 0 \\ (1+g)D_{n-1} + \alpha \max[(A_n - (1+g)D_{n-1}), 0], & n \in \{1, ..., N\}. \end{cases} \quad (11)$$

The inspiration of this contract comes from a pension product in Denmark which is discussed in Guillén, Jørgensen and Nielsen (2006) (GJN's contract). The evolution of the value of customer account in GJN's contract is

$$D'_n = \begin{cases} P, & n = 0 \\ (1+g')D'_{n-1} + \alpha'[A_n - (1+g')D'_{n-1}], & n \in \{1, ..., N\}. \end{cases} \quad (12)$$

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Pricing

Let

$$C_n = \max[(A_n - (1+g)D_{n-1}), 0]. \tag{13}$$

After recursive substitution of equation (11), we get

$$D_N = D_0(1+g)^N + \alpha \sum_{i=1}^{N} C_i(1+g)^{N-i} \tag{14}$$

Then the expected discounted value under risk neutral measure $\mathbb{Q}$ is

$$\begin{aligned} V(0, D_N) &= \tfrac{1}{(1+r_f)^N} E^{\mathbb{Q}}\{D_N\} \\ &= (\tfrac{1+g}{1+r_f})^N D_0 + \tfrac{\alpha}{(1+r_f)^N} E^{\mathbb{Q}}\{\textstyle\sum_{i=1}^{N}[A_i - (1+g)D_{i-1}]_+(1+g)^{N-i}\} \end{aligned} \tag{15}$$

where $r_f = e^r - 1$

Pricing

The relationship between guarantee rate g and participation rate α is presented in the following figure.

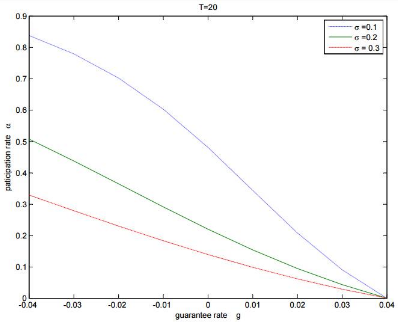

Figure: $T = 20$ years, $r_f = 0.04$.

Results

In order to examine if our new contract outperforms GJN's contract, 100,000 equity index price paths are simulated under real world measure $\mathbb{P}$.

	GJN	New Contract	Bond	Equity Index
Expected wealth	3.0612	2.8793	2.1911	3.5243
Std of wealth	1.6829	1.4889	0	2.6340
Expected CPT Utility	1.2816	1.3330	1.0914	1.2785

Table: The mean, standard deviation and expected CPT utility of holding each asset. ($\alpha = 0.1286$. $T = 20$ years, $r_f = 0.04$, $\mu = 0.065$, $g = 0.02$, $g' = 0.04$ and $\sigma = 0.15$, $P = 1$.)

Results

If the customer can freely choose their portfolio, then the proportion of each asset ($\omega_i \in [0,1], \sum \omega_i = 1$) in the optimal portfolio under CPT is:

	GJN	New Contract	Bond	Equity Index
weight	0	0.61	0	0.39

Table: The optimal portfolio under CPT. ($\alpha = 0.1286$. $T = 20$ years, $r_f = 0.04$, $\mu = 0.065$, $g = 0.02$, $g' = 0.04$ and $\sigma = 0.15, P = 1$.)

CPT utility of this optimized portfolio is 1.3747 which is larger than holding any one asset.

Results

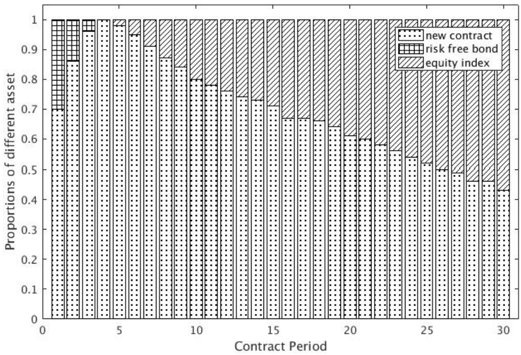

Figure: The composition of optimal portfolio for different terms. $g = 0.02$, $r_f = 0.04$, $\mu = 0.065$ and $\sigma = 0.15$.

Results

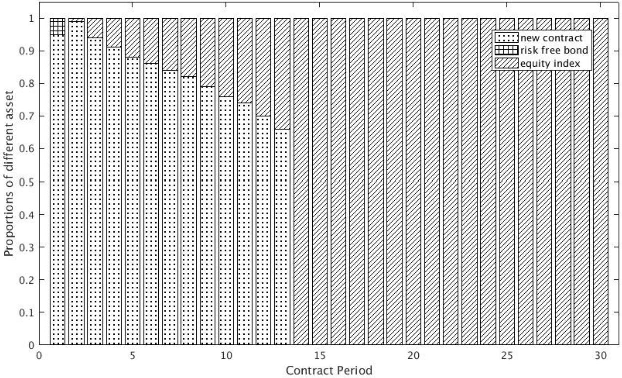

Figure: The composition of optimal portfolio for different terms. $g = 0.02$, $r_f = 0.04$, $\mu = 0.1$ and $\sigma = 0.2$.

Results

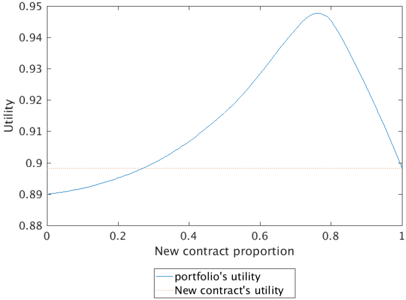

Figure: The composition of optimal portfolio for term of $T = 10$.

Results

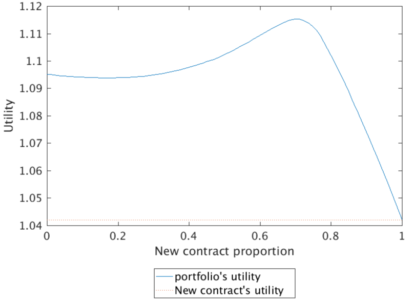

Figure: The composition of optimal portfolio for term of $T = 12$.

Results

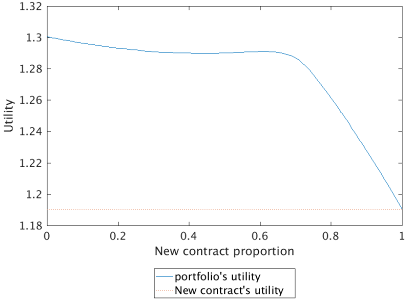

Figure: The composition of optimal portfolio for term of $T = 14$.

Results

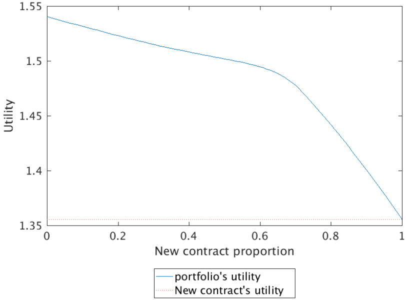

Figure: The composition of optimal portfolio for term of $T = 16$.

Conclusion

- In this paper, we introduce a new pension contract with the features of guarantees and bonuses. It has transparent structure and clear distribution rule.
- Under cumulative prospect theory, the contract generates higher utility than the contract introduced in Guillén, Jørgensen and Nielsen (2006). The result provides the evidence why the guarantees should be included in the pension contract.
- In addition, our result shows with the increase of policyholder's investment horizons, the proportion of risky asset in optimal portfolio also increases while the proportion of risk free asset decreases. This result conforms to the traditional pension investment advice.

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