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Methods of pooling longevity risk

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http://risk-insight-lab.com

The **'Minimising Longevity and Investment Risk while Optimising Future Pension Plans'** research programme is being funded by the Actuarial Research Centre.

22 May 2018

Overview of entire session

I. Motivation

II. One way of pooling longevity risk

III. Classification of methods & discussion

IV. A second explicit scheme

V. An implicit scheme

VI. Summary and discussion

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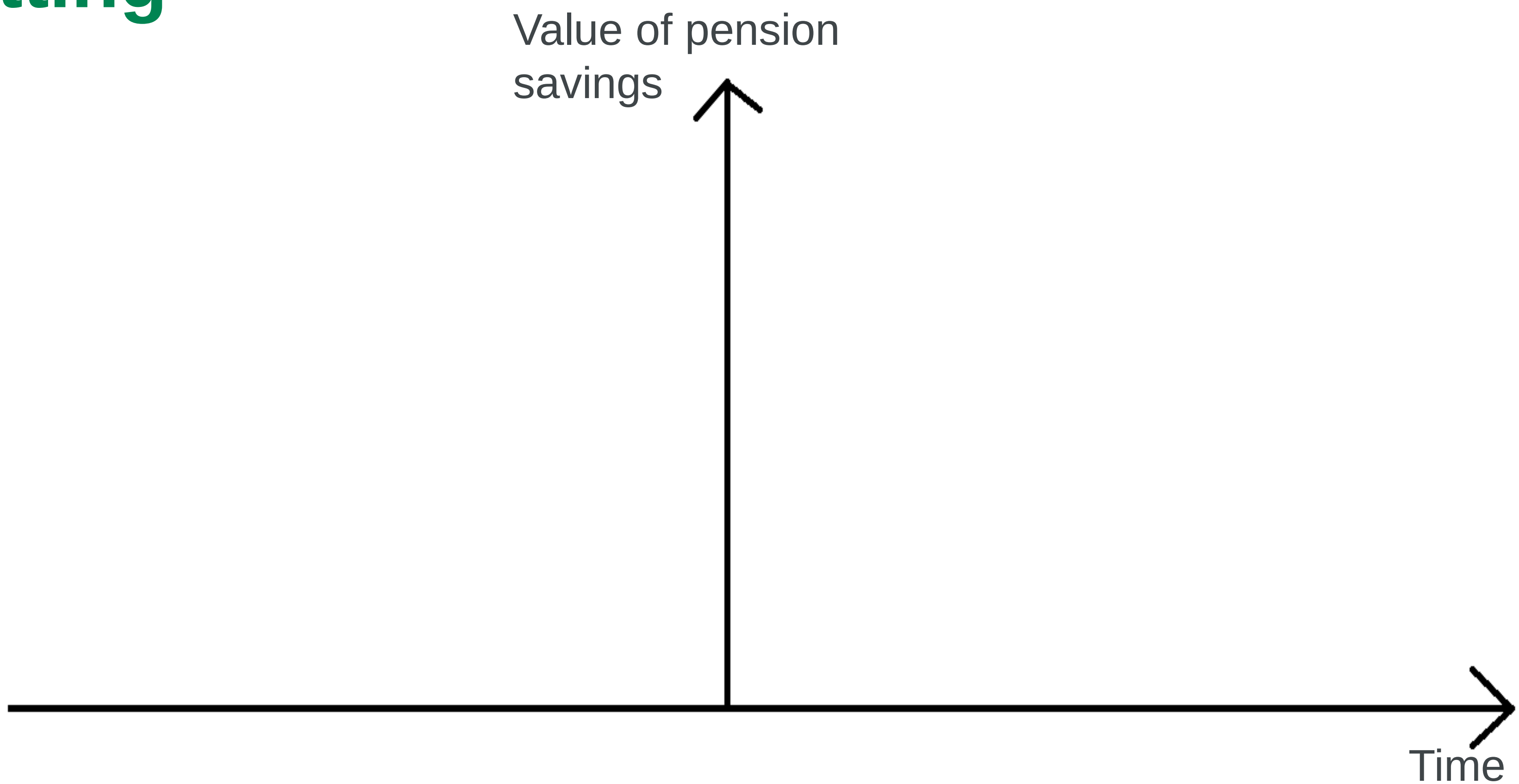
I. Motivation

- Background
- Focus on life annuity
- Example of a tontine in action

Setting



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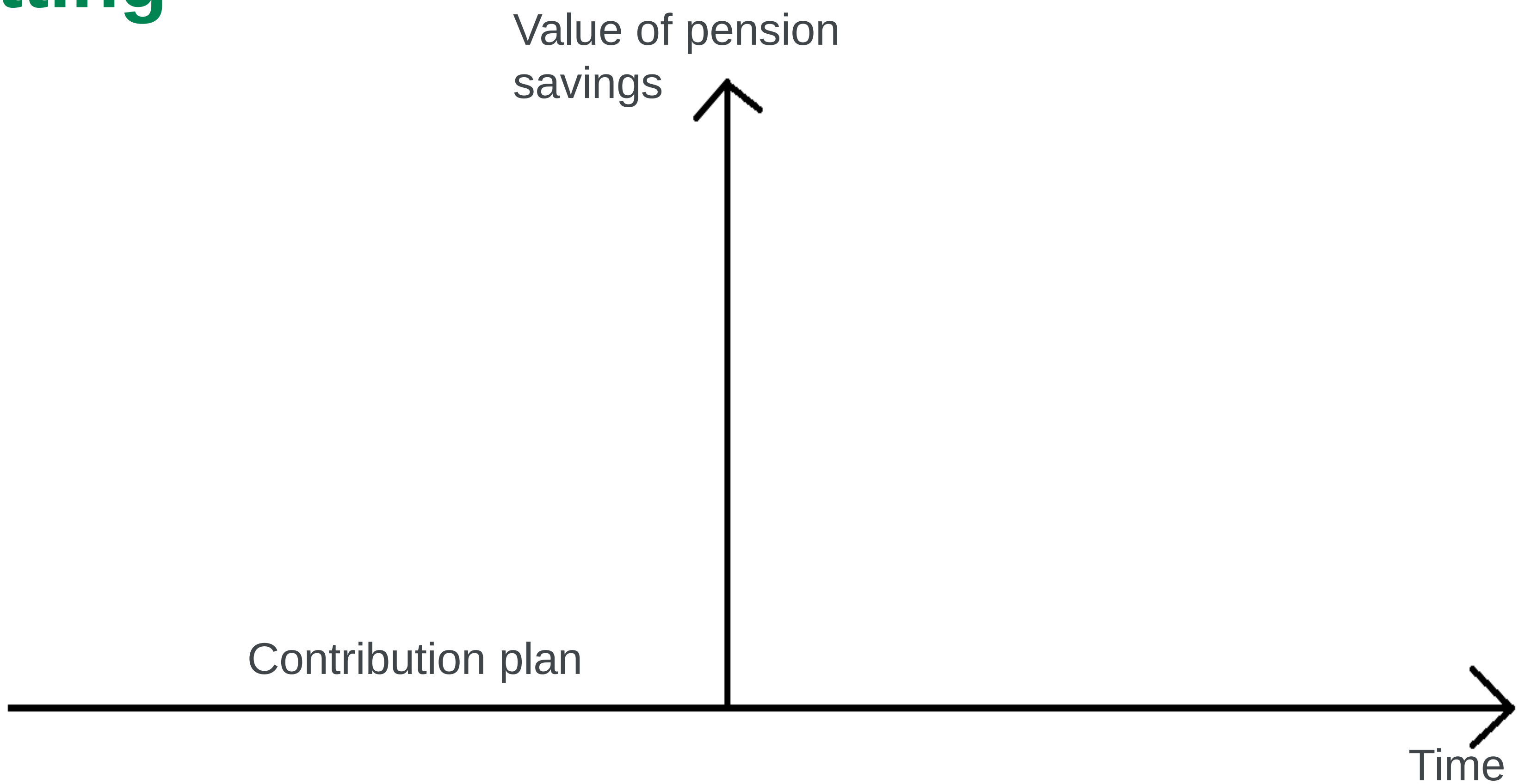

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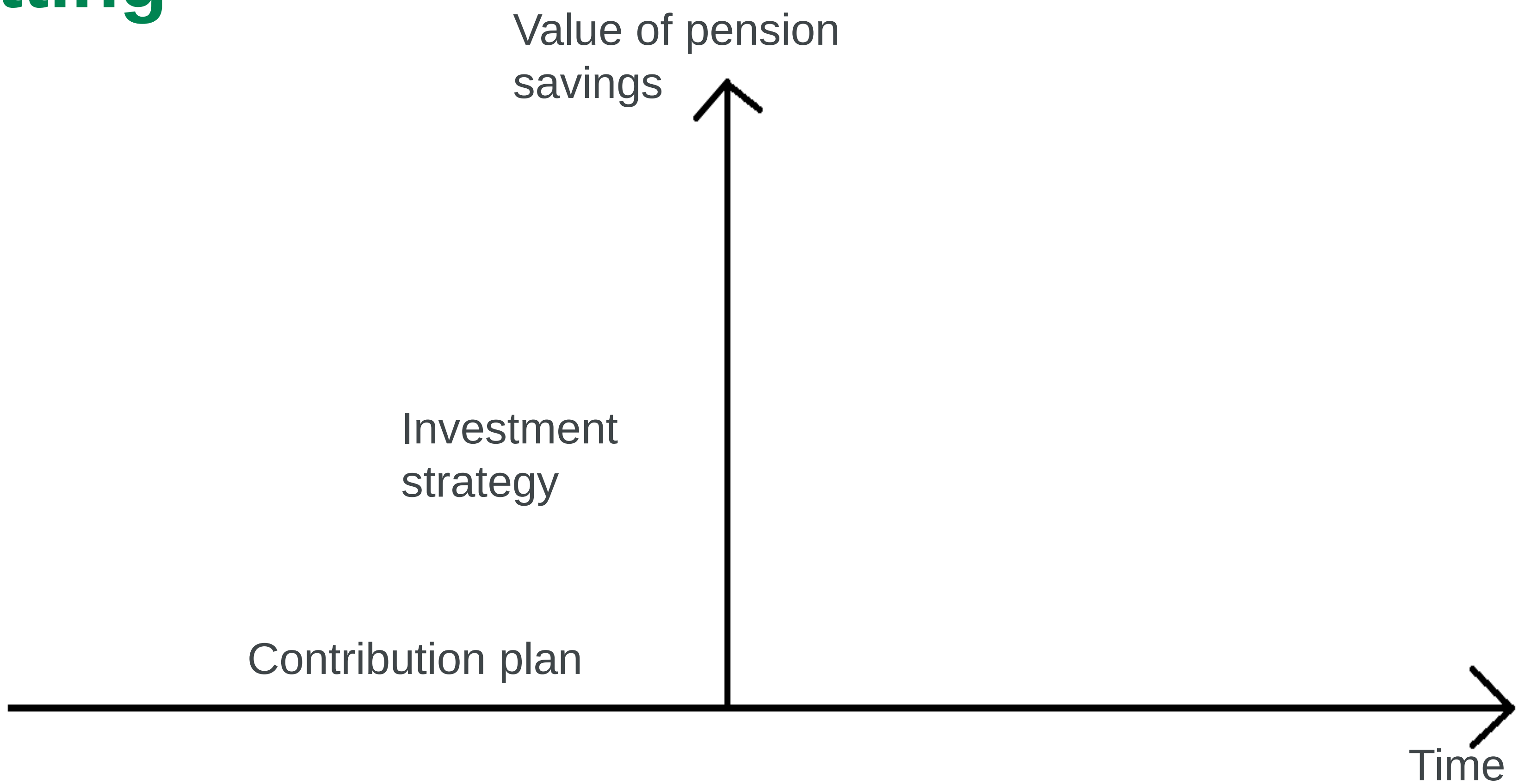
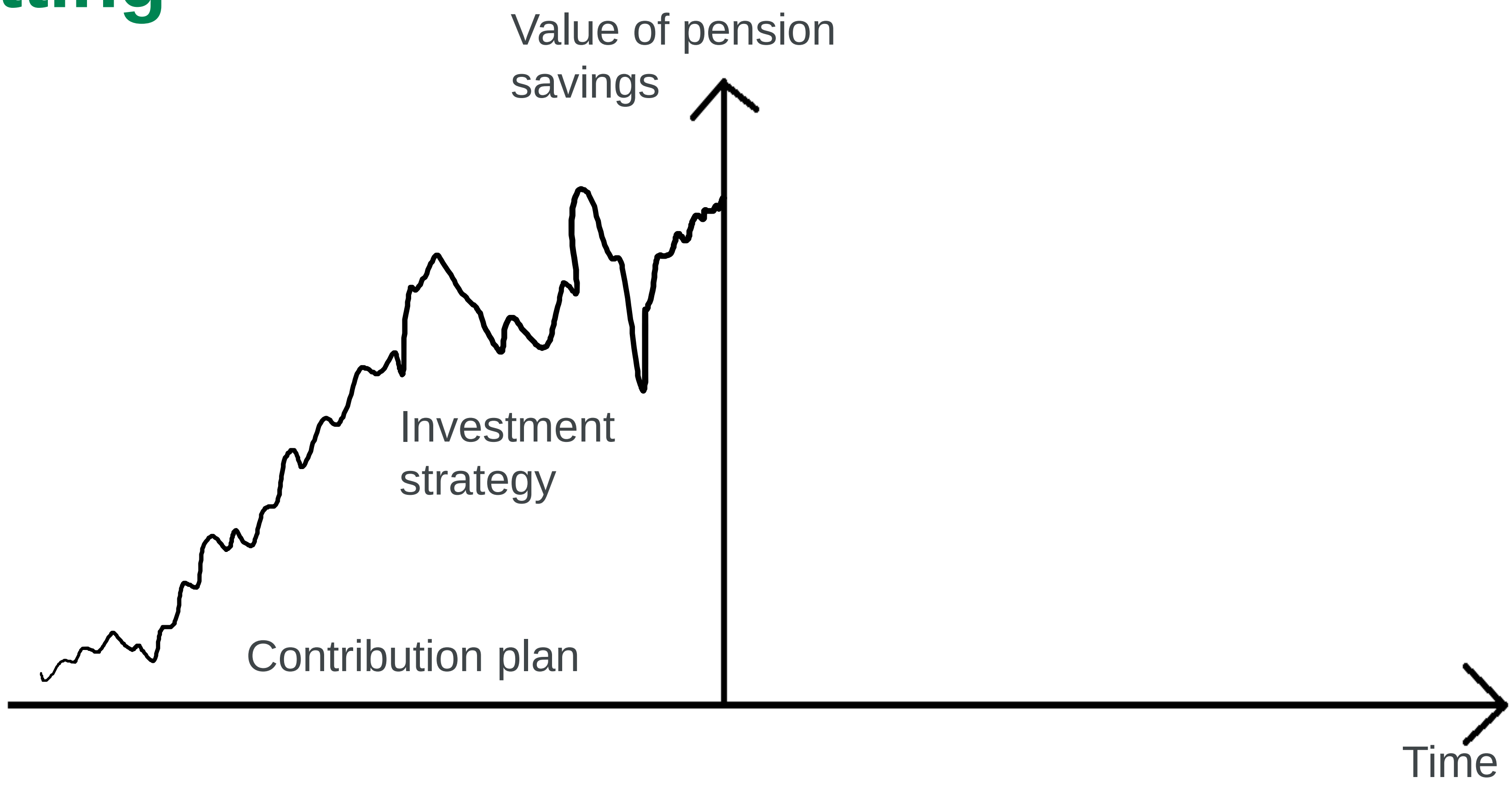

Value of pension savings
Investment strategy
Contribution plan
Time

Setting



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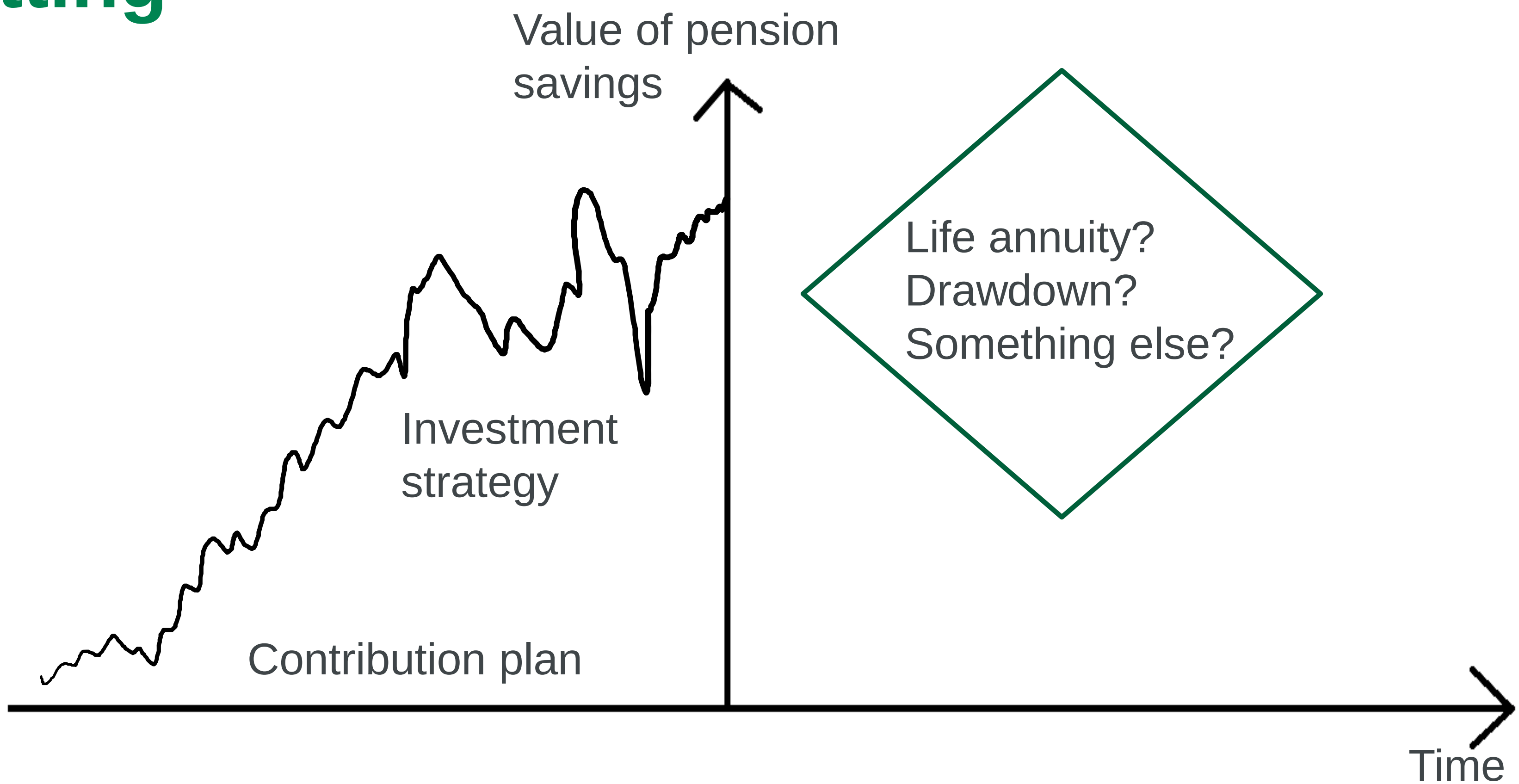

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The present in the UK – DC on the rise

- Defined benefit plans are closing (87% are closed in 2016 in UK).

- Most people are now actively in defined contribution plans, or similar arrangement (97% of new hires in FTSE350).

- Contribution rates are much lower in defined contribution plans

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Size of pension fund assets in 2016
[Willis Towers Watson]

Country	Value of pension fund assets (USD billion)	As percentage of GDP	Of which DC asset value (USD billion)
USA	22’480	121.1%	13’488
UK	2’868	108.2%	516
Japan	2’808	59.4%	112
Australia	1’583	126.0%	1’377
Canada	1’575	102.8%	79
Netherlands	1’296	168.3%	78

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Drawdown

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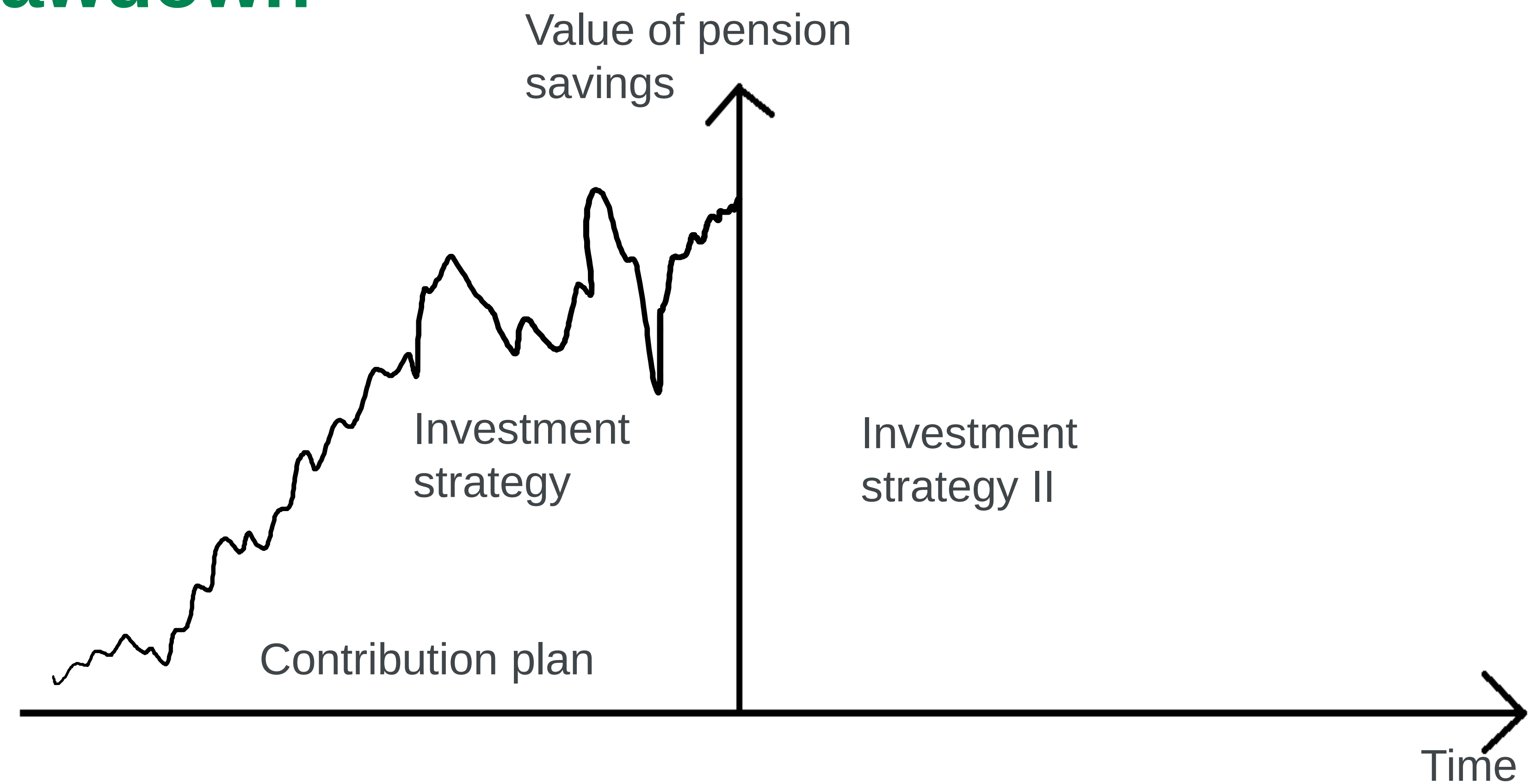
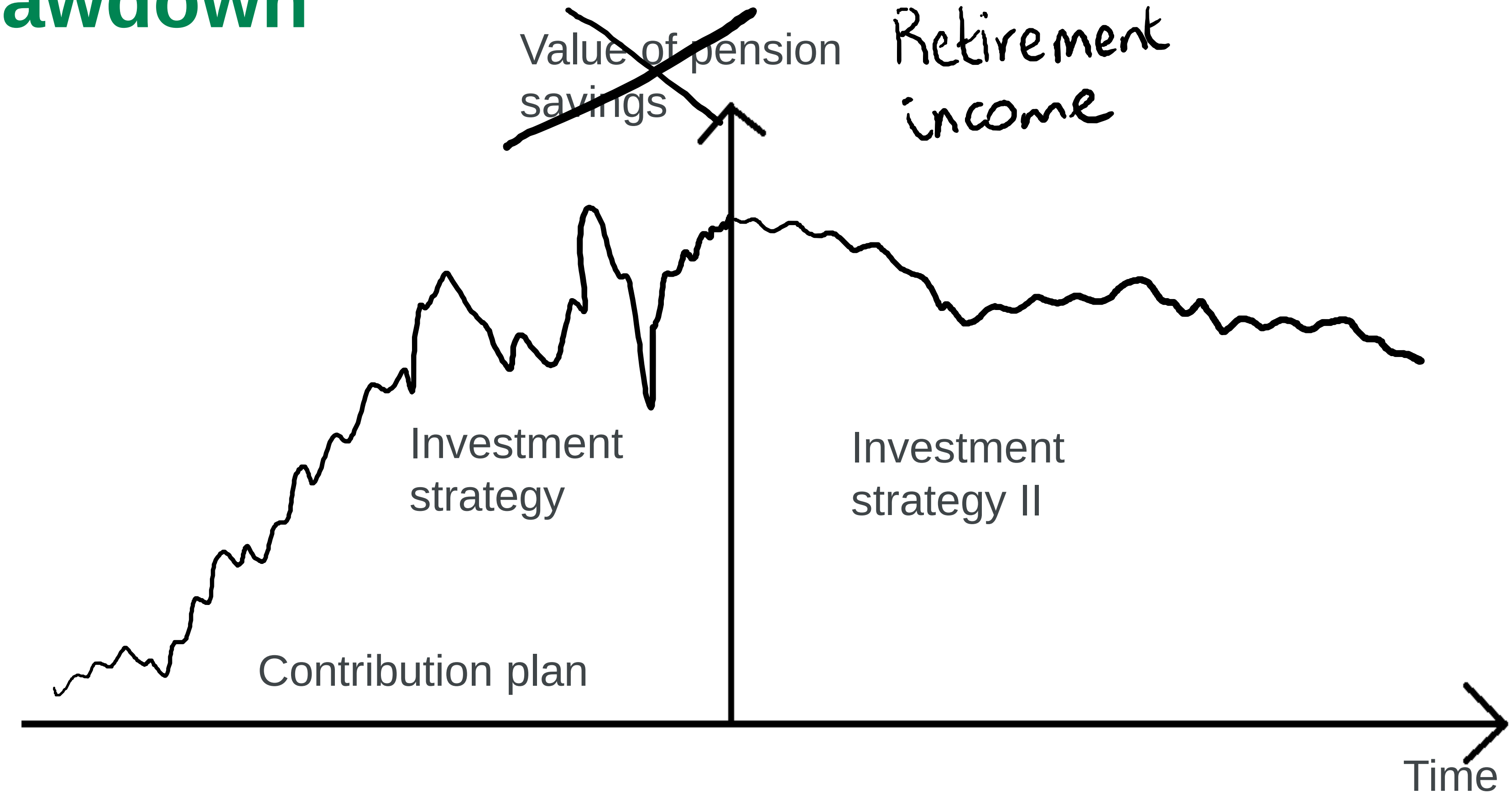
Drawdown



Value of pension savings
Investment strategy
Investment strategy II
Contribution plan
Time

Drawdown


Value of pension savings
Retirement income
Investment strategy
Investment strategy II
Contribution plan
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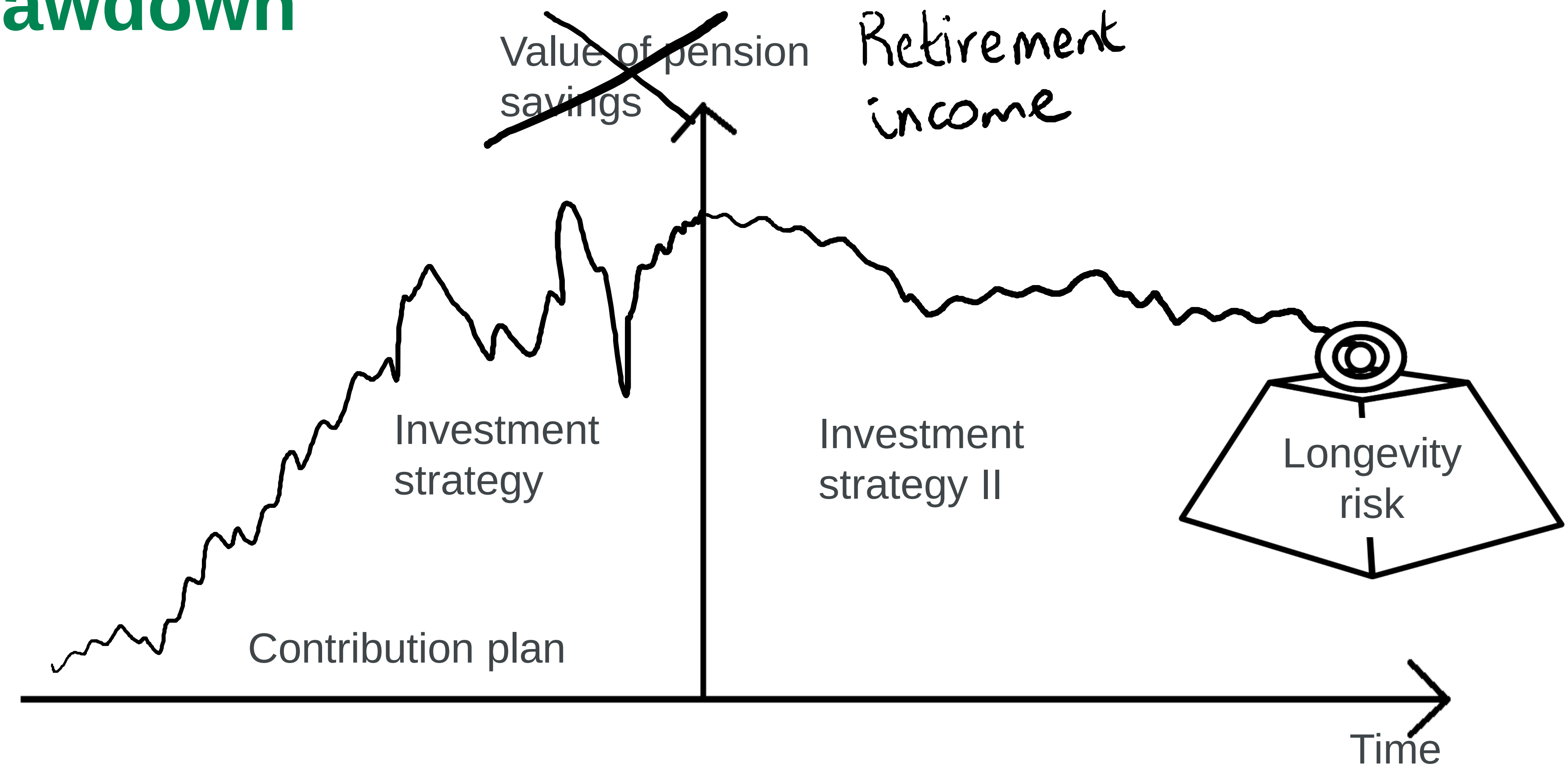
Drawdown



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Life insurance mathematics 101

- PV(annuity paid from age 65) = $a_{\overline{T|}}$

- Expected value of the PV is

$$a_{65} = vp_{65} + v^2\,{}_2p_{65} + v^3\,{}_3p_{65} + v^4\,{}_4p_{65} + \cdots$$

- To use as the price,
  - Law of Large Numbers holds,
  - Same investment strategy,
  - Known investment returns and future lifetime distribution.

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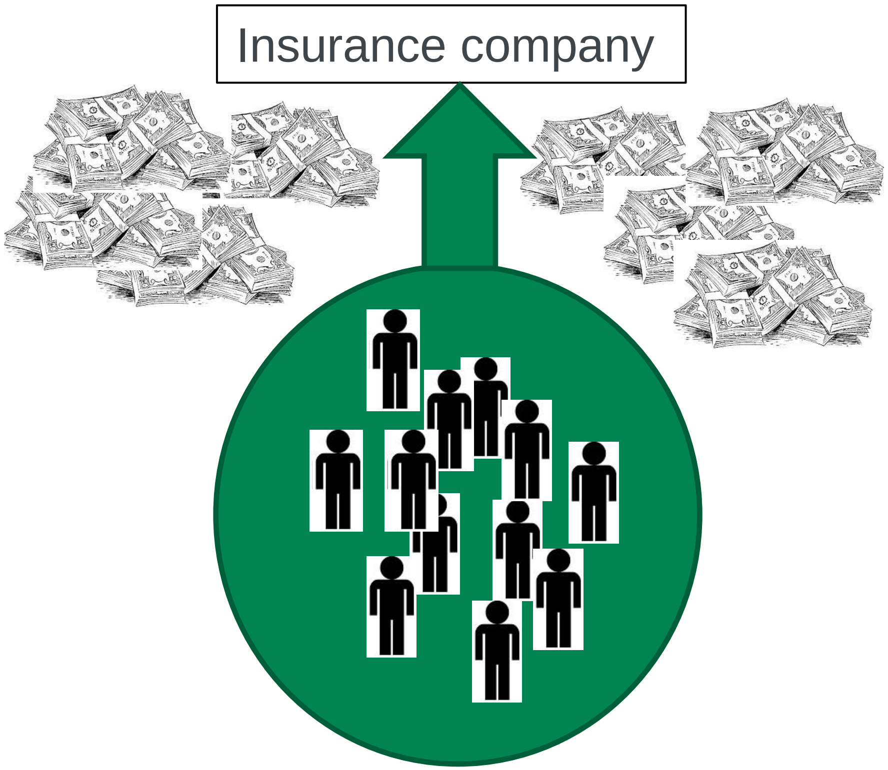
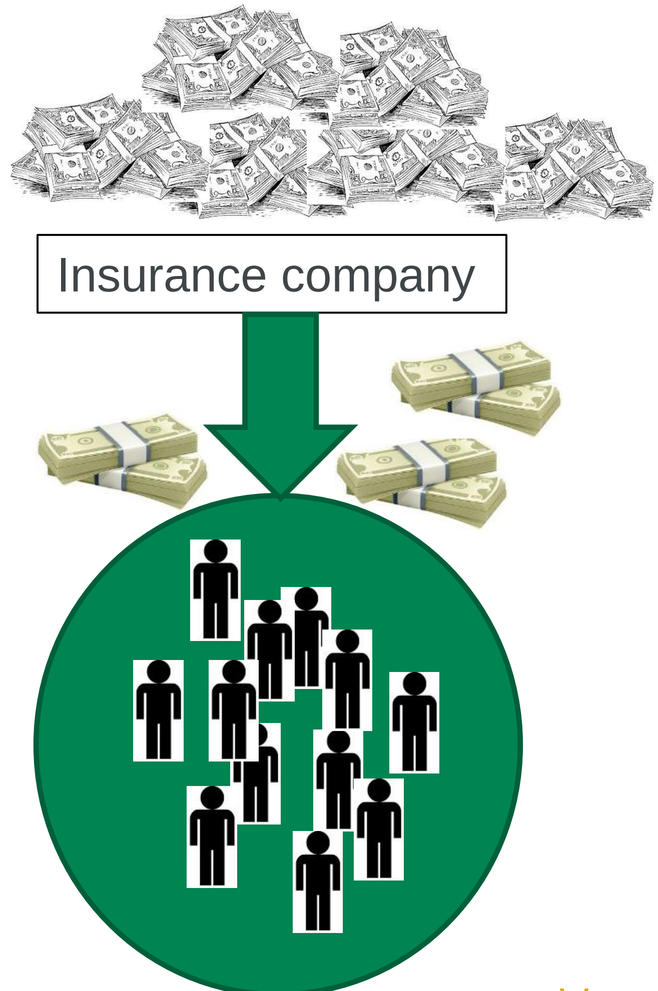
Life annuity contract



Purchase of the annuity contract



Annuity income

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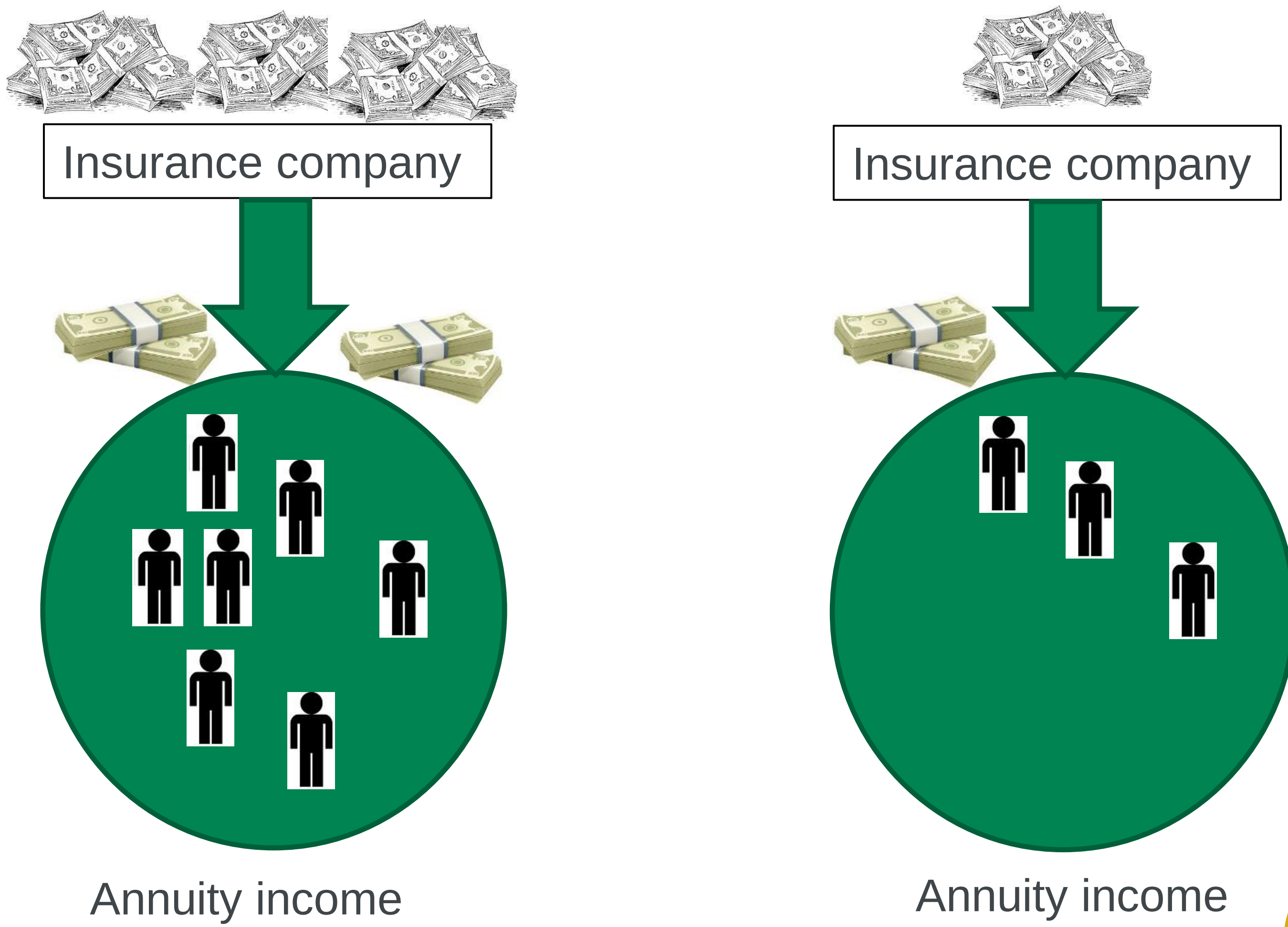
Life annuity contract



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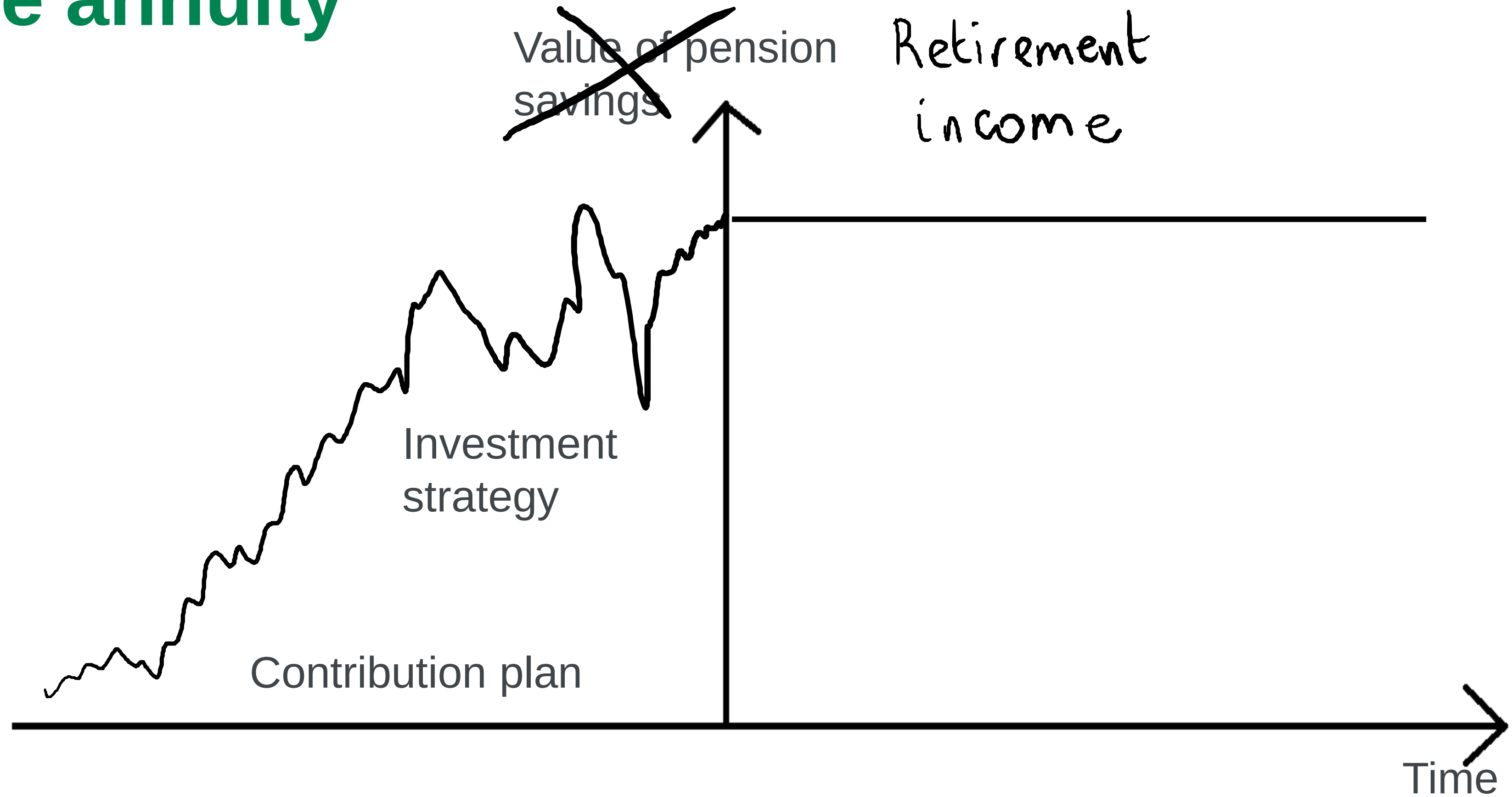
Life annuity



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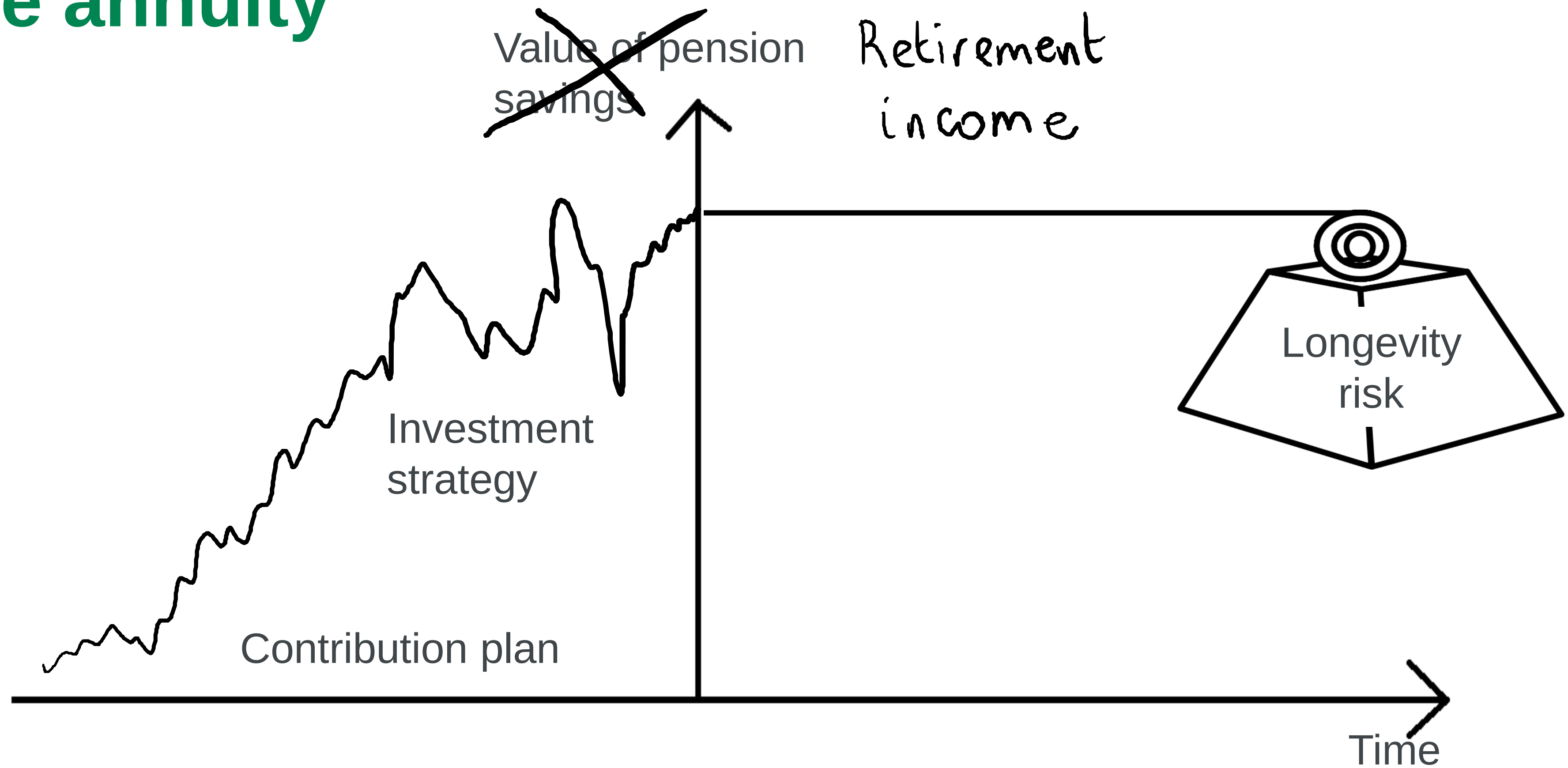
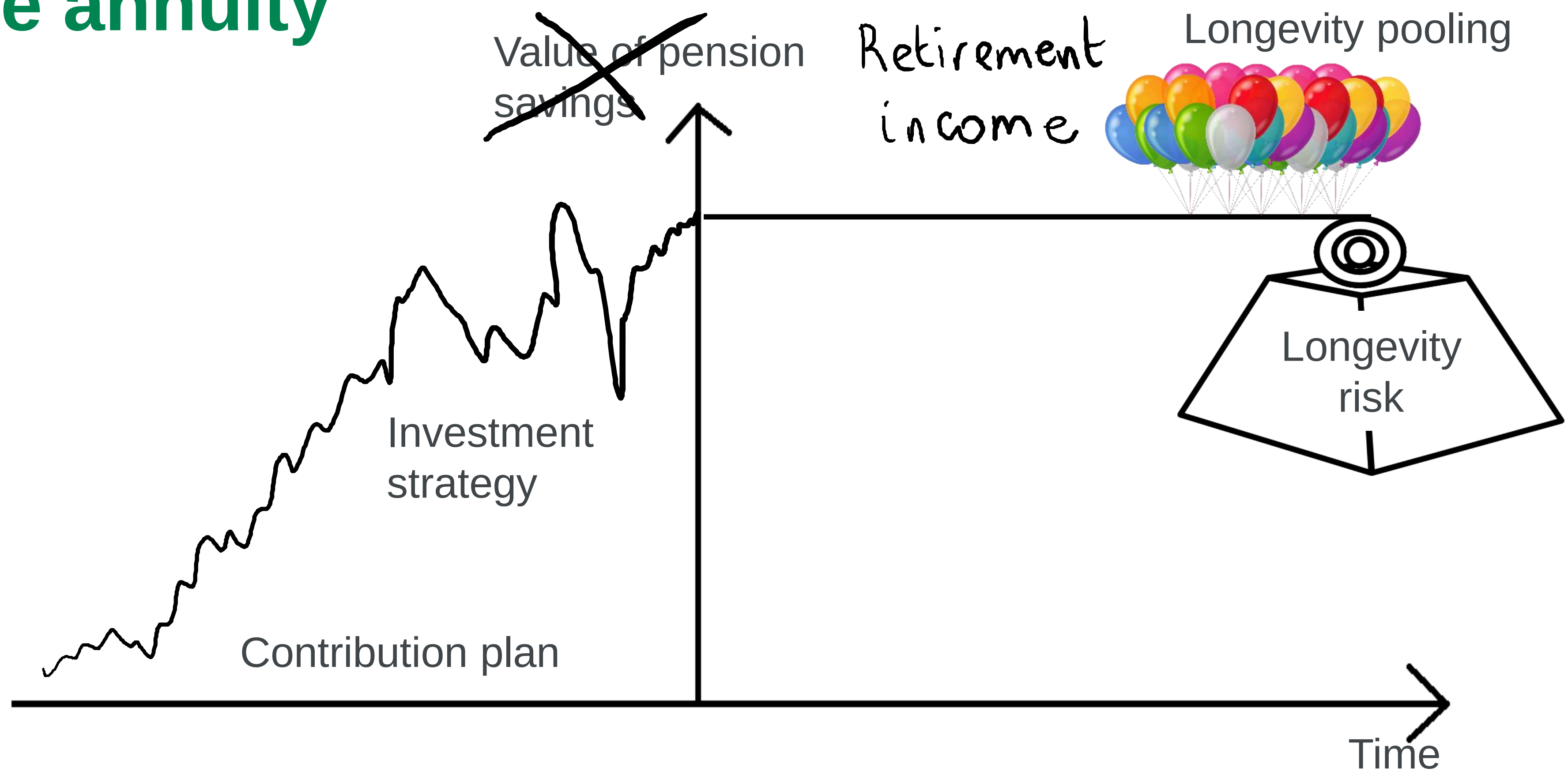
Life annuity



Life annuity



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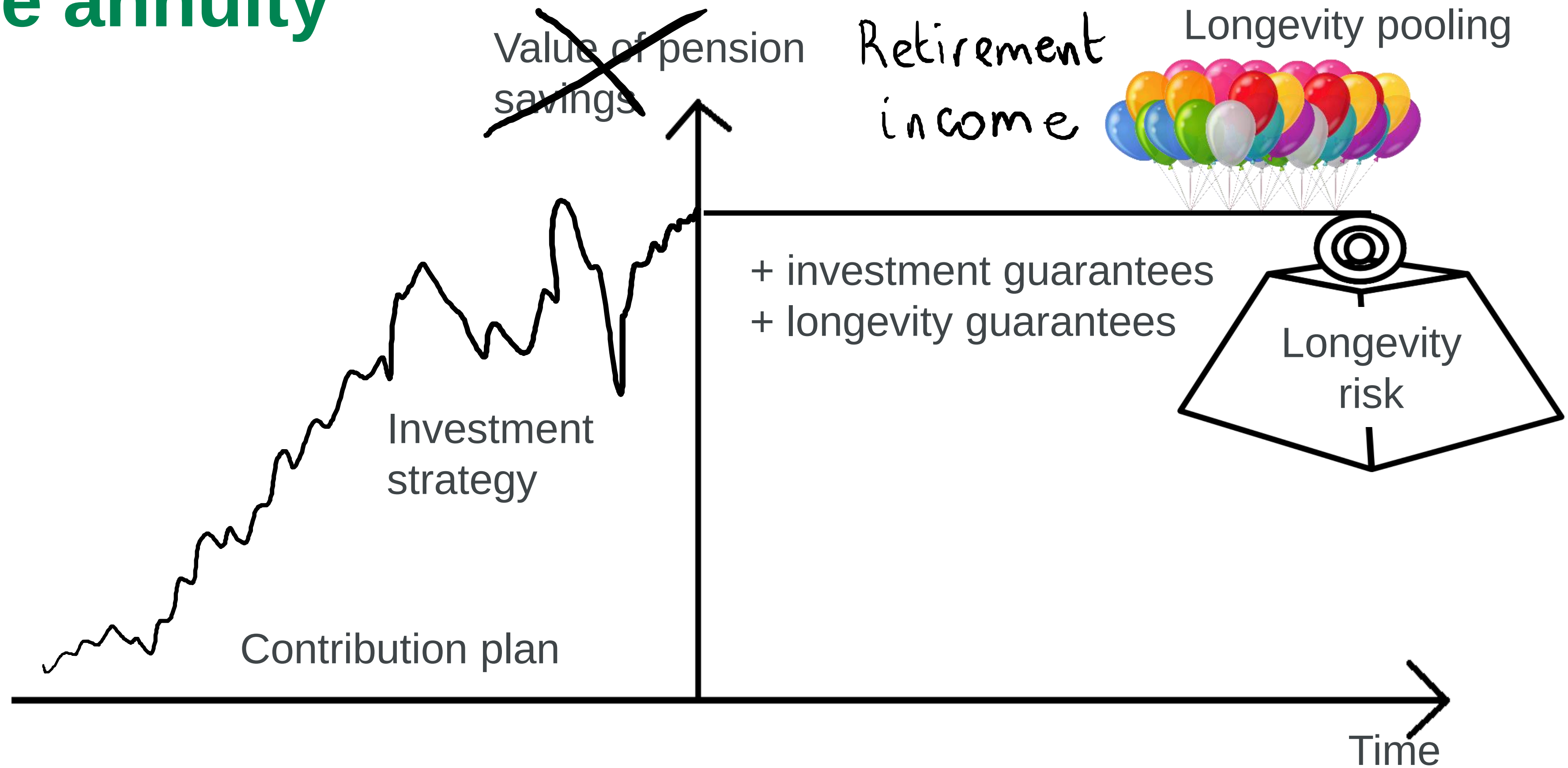
Life annuity



Life annuity contract

- Income drawdown vs life annuity: if follow same investment strategy then life annuity gives higher income*

*ignoring fees, costs, taxes, etc.

- Pooling longevity risk gives a higher income.

- Everyone in the group becomes the beneficiaries of each other, indirectly.

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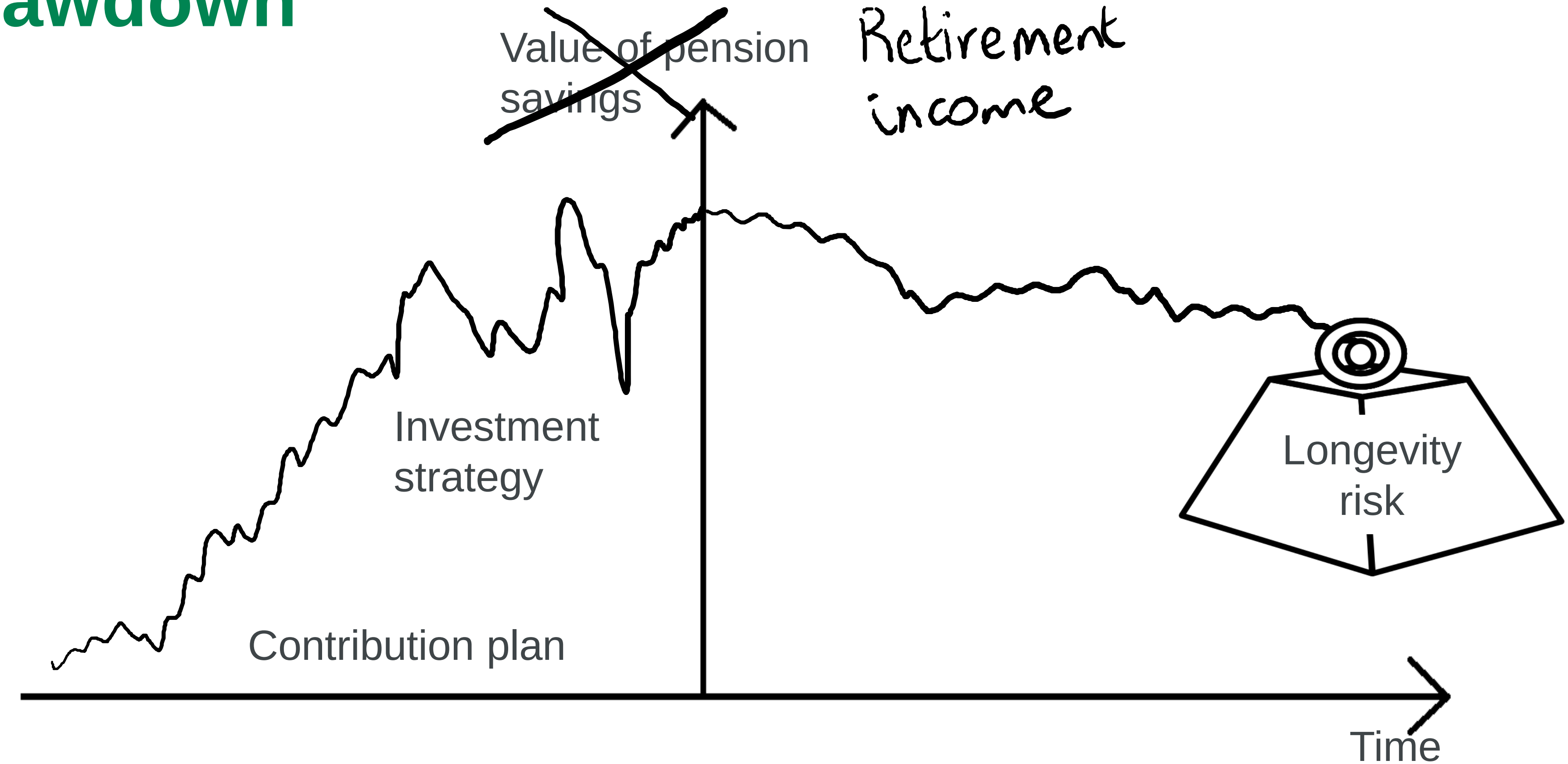
Annuity puzzle

- Why don’t people annuitize?

- Can we get the benefits of life annuities, without the full contract?

- Example showing income withdrawal from a tontine.

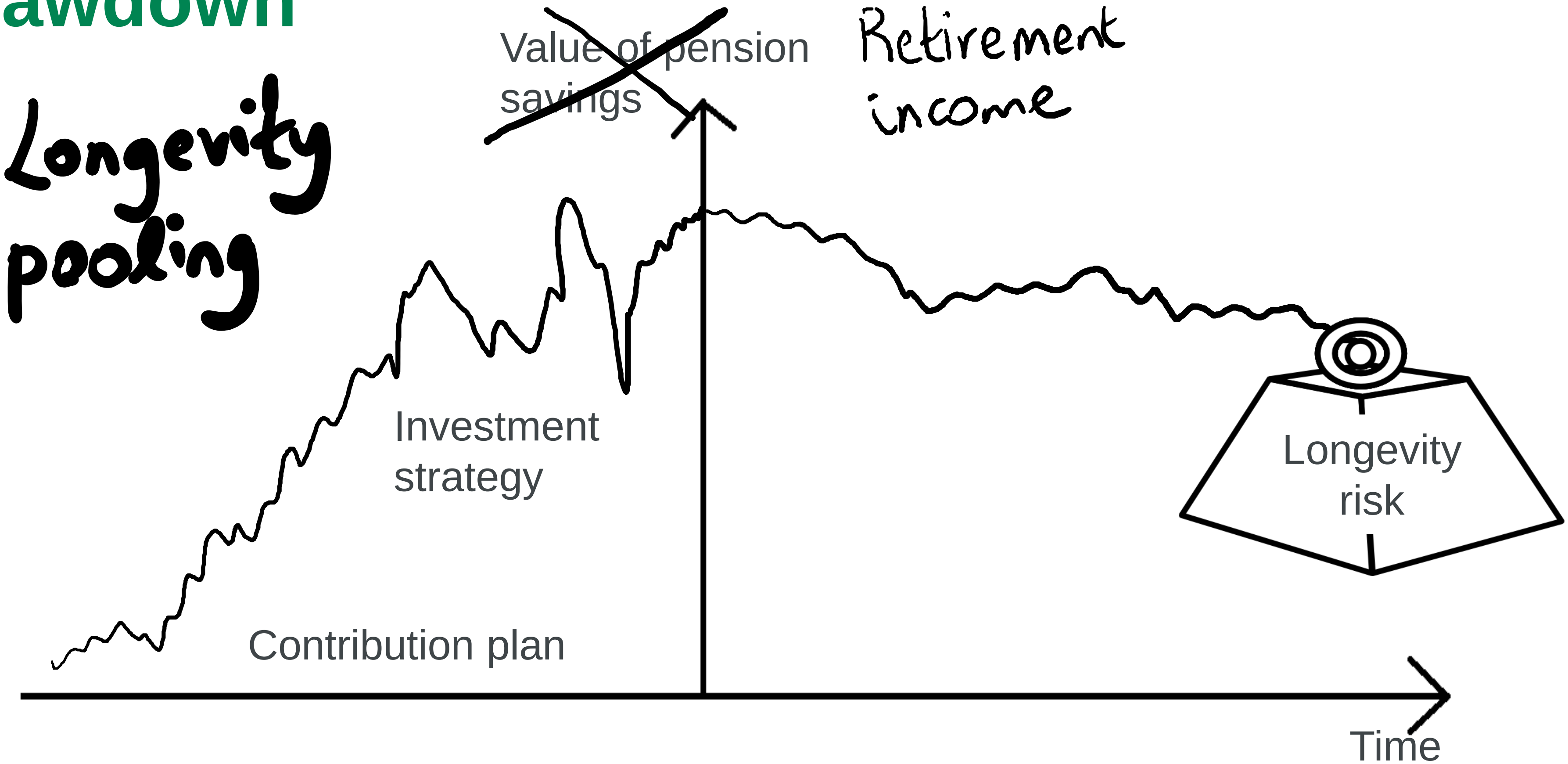
Drawdown



Value of pension savings
Retirement income
Investment strategy
Contribution plan
Longevity risk
Time

Drawdown

+ Longevity pooling



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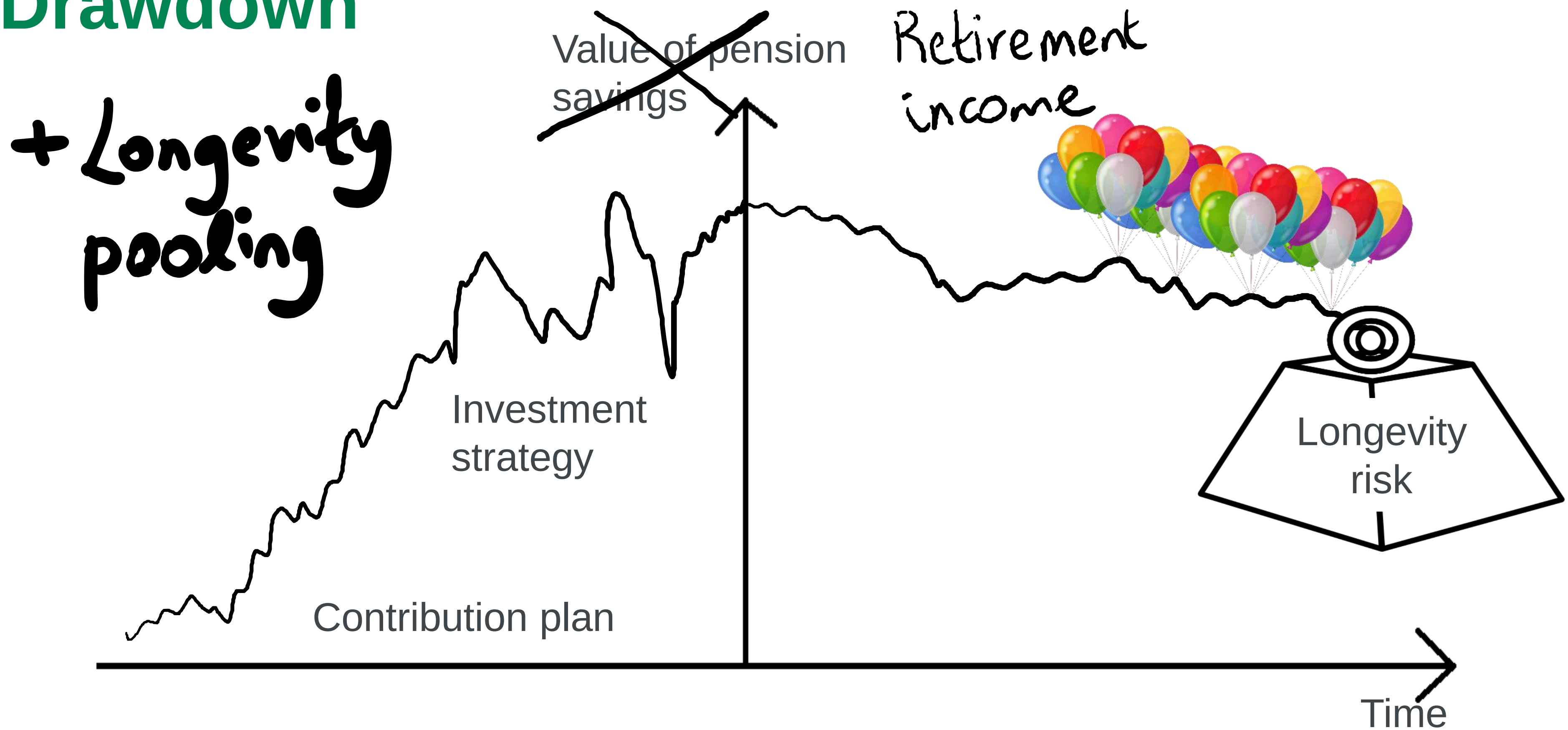
Drawdown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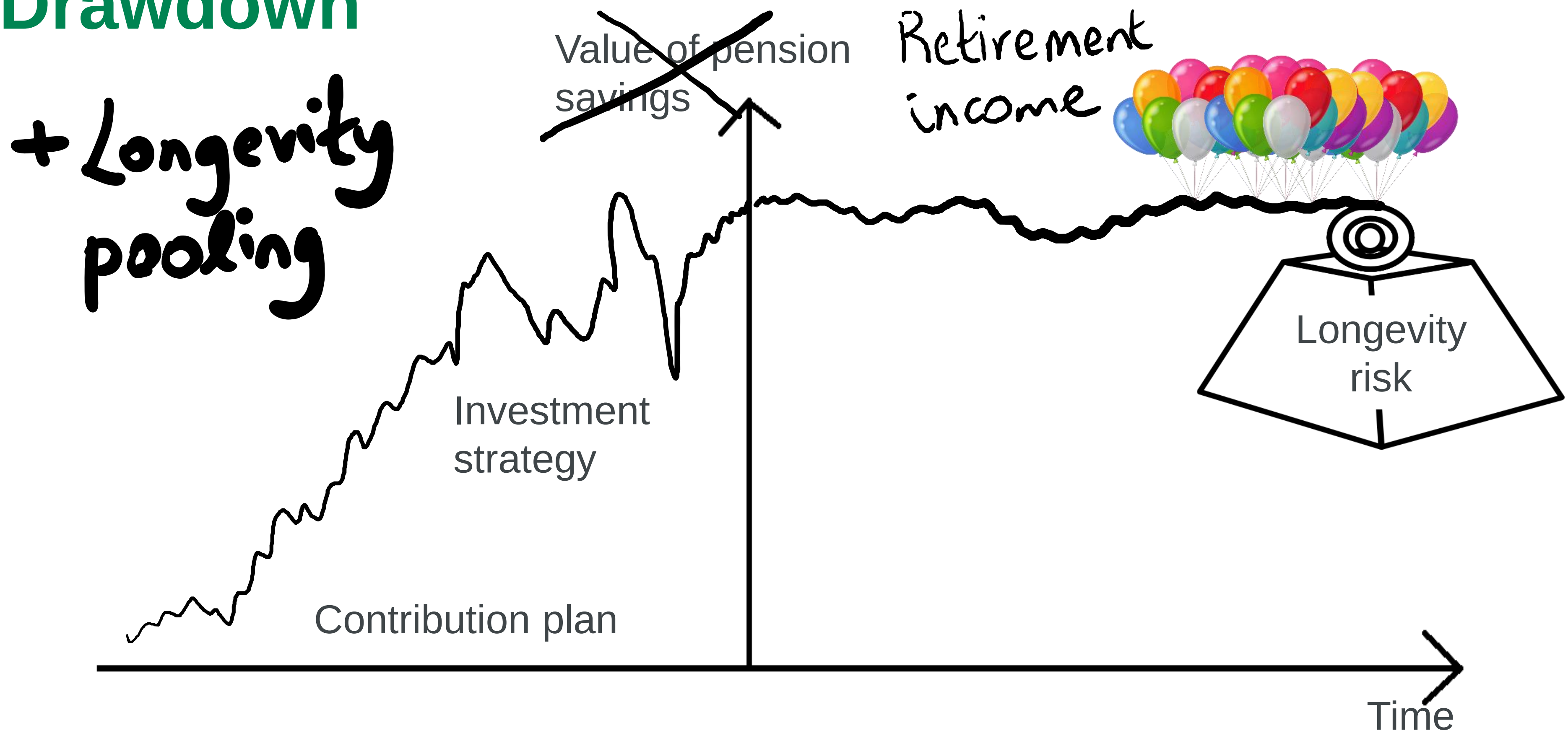

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Aim of modern tontines

- Aim is to provide an income for life.
- It is not about gambling on your death or the deaths of others in the pool.
- It should look like a life annuity.
- With more flexibility in structure.
- Example is based on an explicitly-paid longevity credit.

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Example 0: Simple setting of 4% Rule

- Pension savings = €100,000 at age 65.
- Withdraw €4,000 per annum at start of each year until funds exhausted.
- Investment returns = Price inflation + 0%.
- No longevity pooling.

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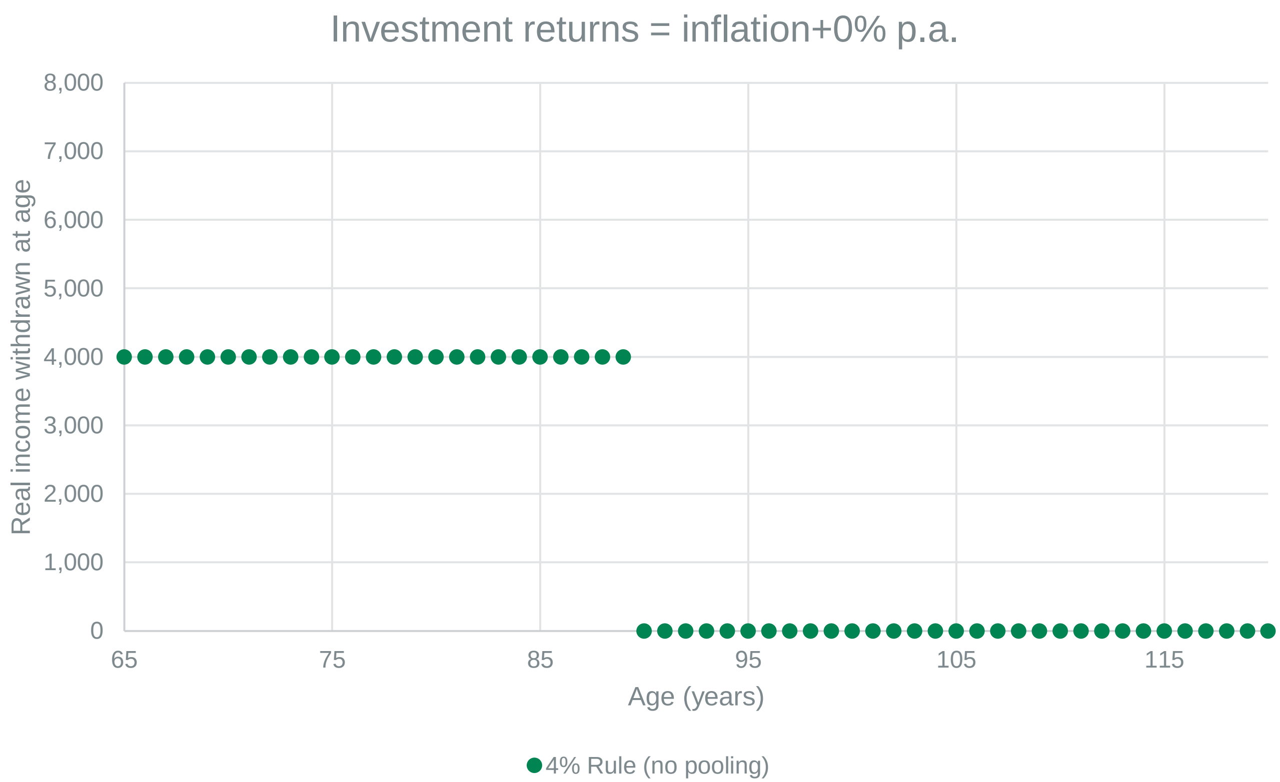
Example 0: income drawdown (4% Rule)



Example 1: Join a tontine

- Same setup except…pool all of asset value in a tontine for rest of life.

- Withdraw a maximum real income of €*X* per annum for life

(we show *X* on charts to follow).

- Mortality table S1PMA.

- Assume a perfect pool: longevity credit=its expected value.

- Longevity credit paid at start of each year.

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UK mortality table S1PMA

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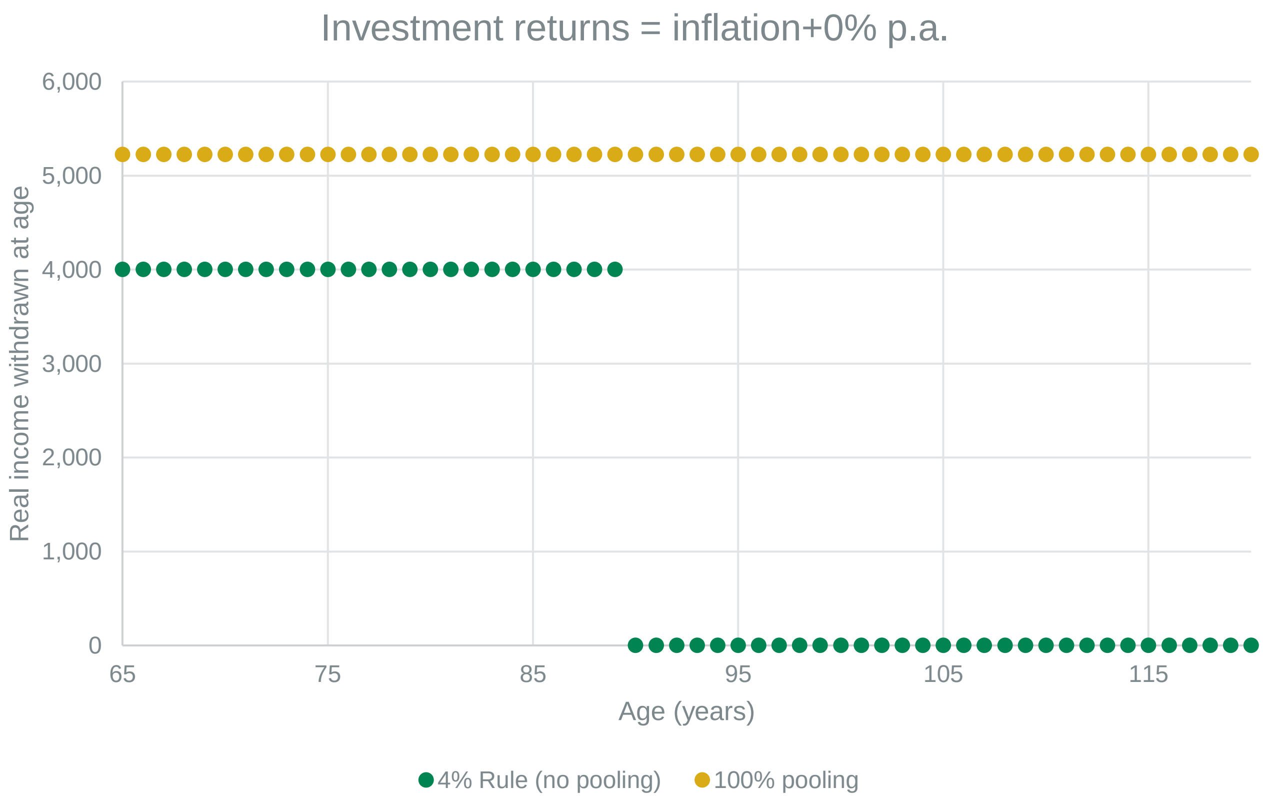
Example 1i:
0% investment returns above inflation



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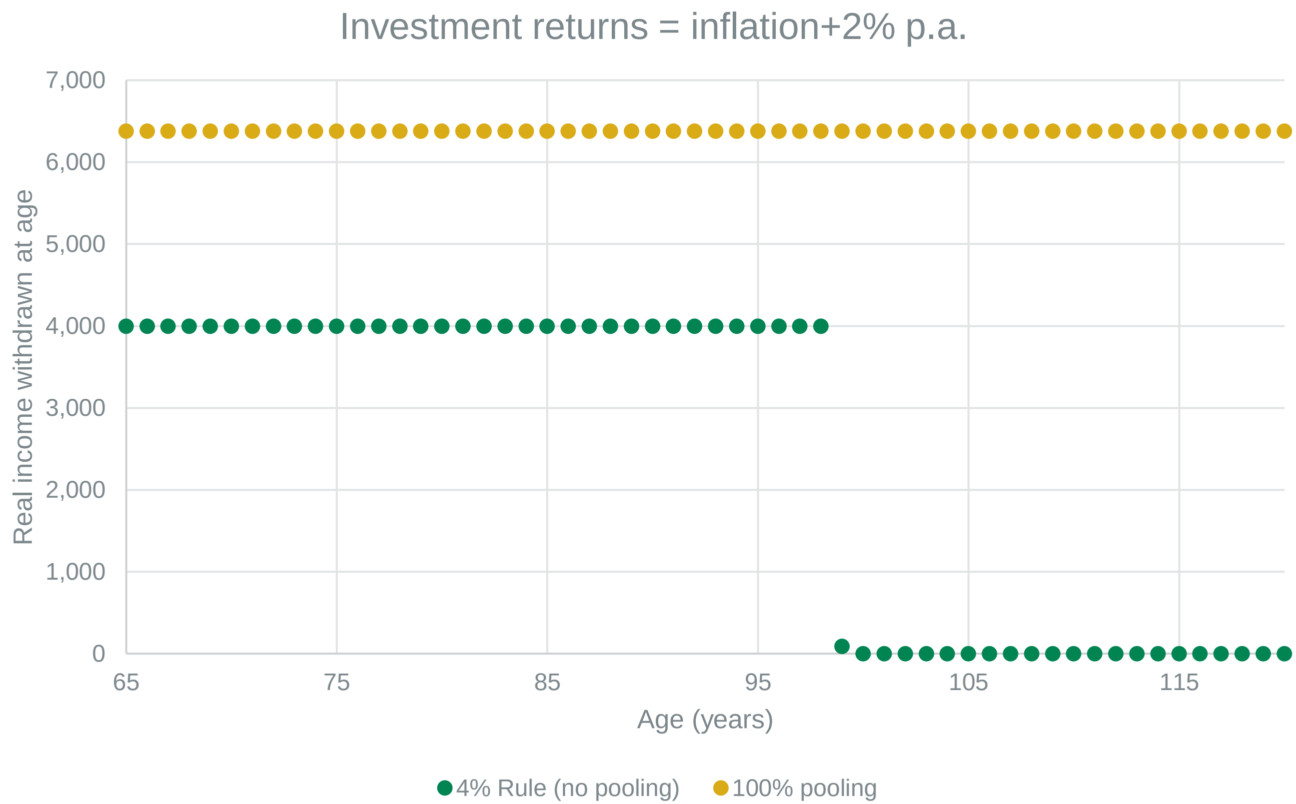
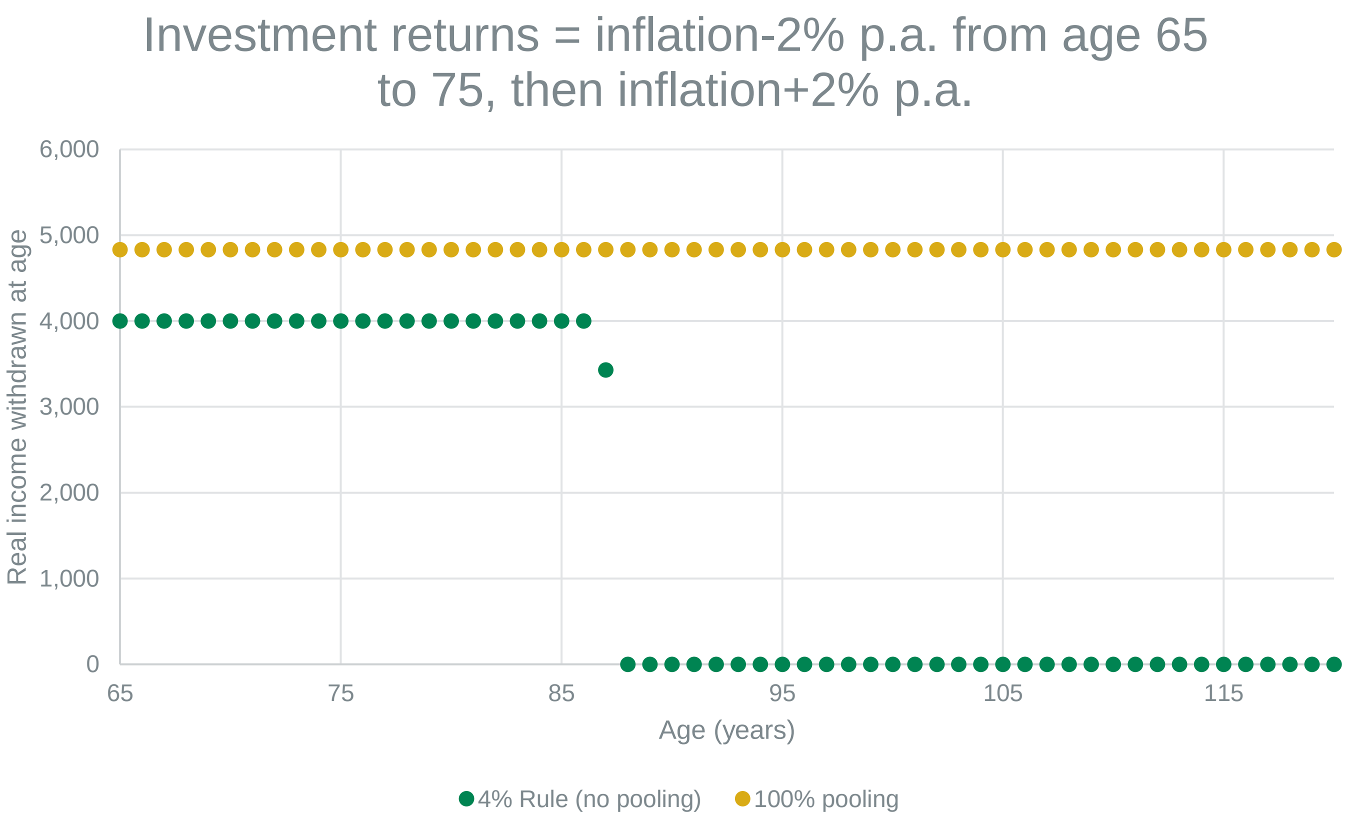
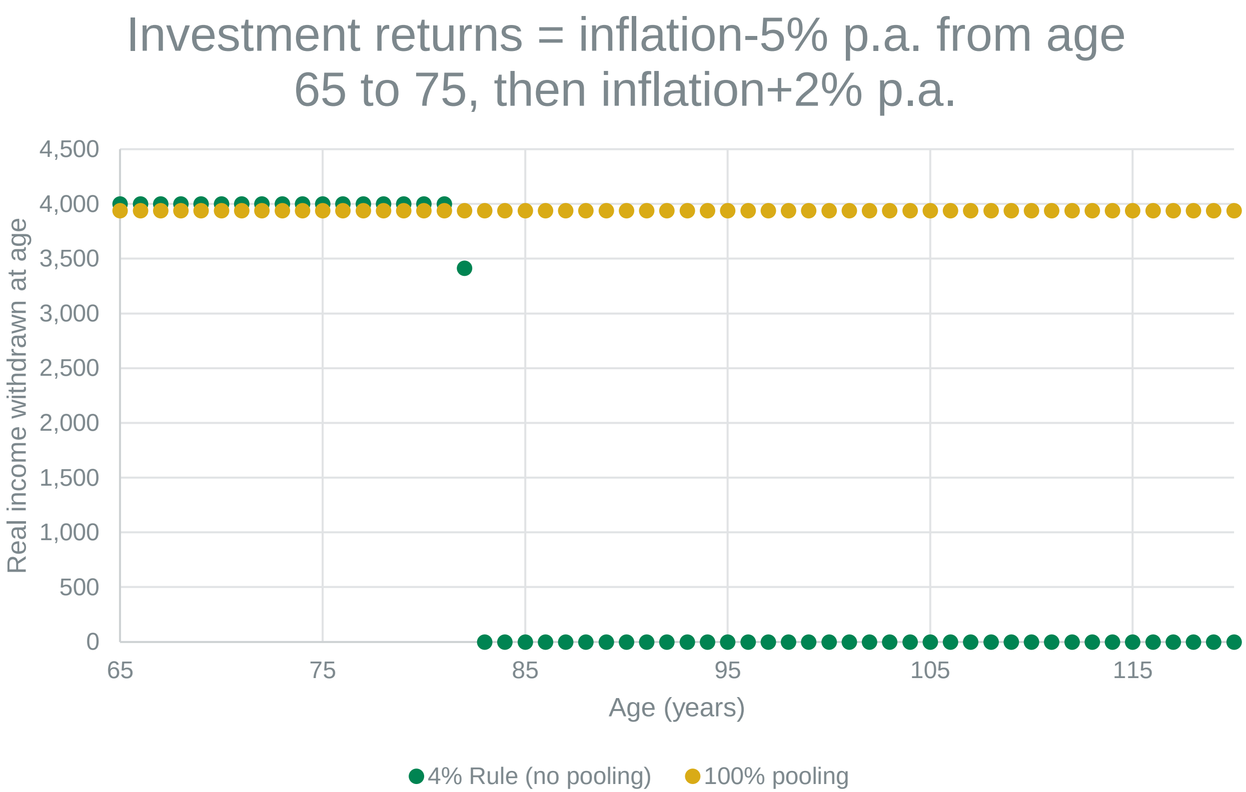
Example 1ii:
+2% p.a. investment returns above inflation



Example 1iii: Inv. Returns = Inflation – 2% p.a. from age 65 to 75, then Inflation +2% p.a.



Example 1iv: Inv. Returns = Inflation – 5% p.a. from age 65 to 75, then Inflation +2% p.a.



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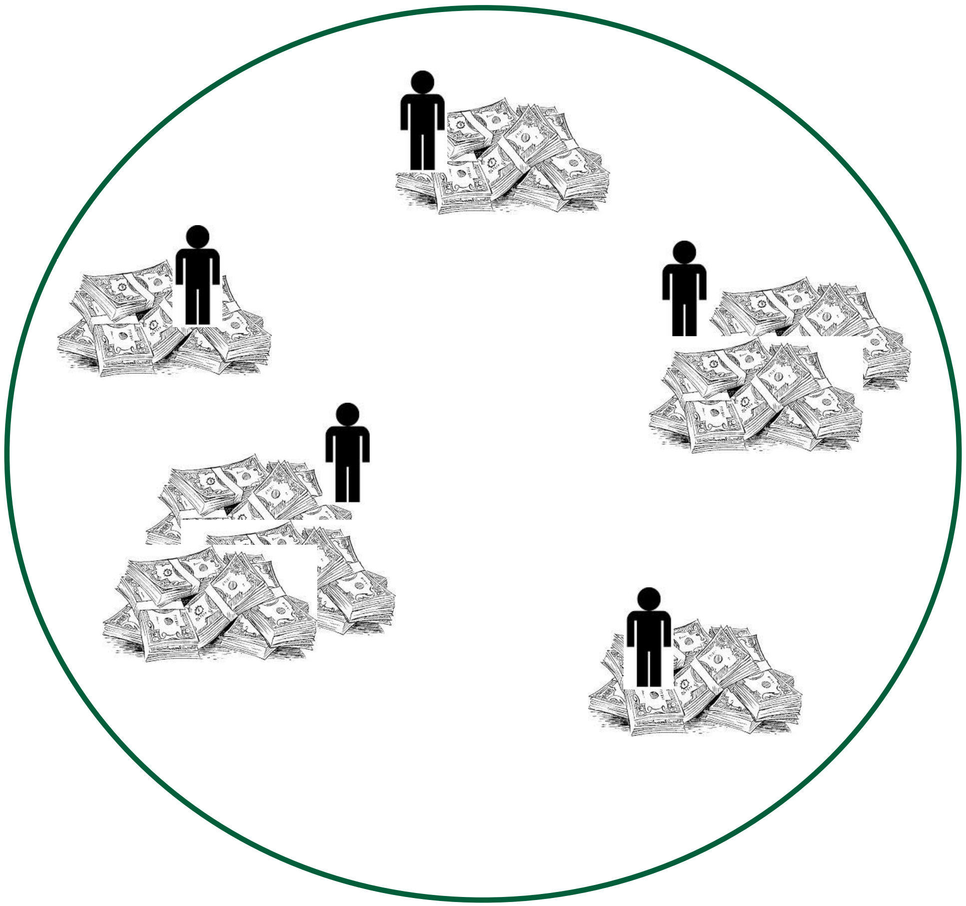
II. One way of pooling longevity risk

- Aim of pooling: retirement income, not a life-death gamble.

- DGN method of pooling longevity risk
  - Explicit scheme.
  - Everything can be different: member characteristics, investment strategy.

Longevity risk pooling



Pool risk over lifetime

Individuals make their own investment decisions

Individuals withdraw income from their own funds

However, when someone dies at time *T*…

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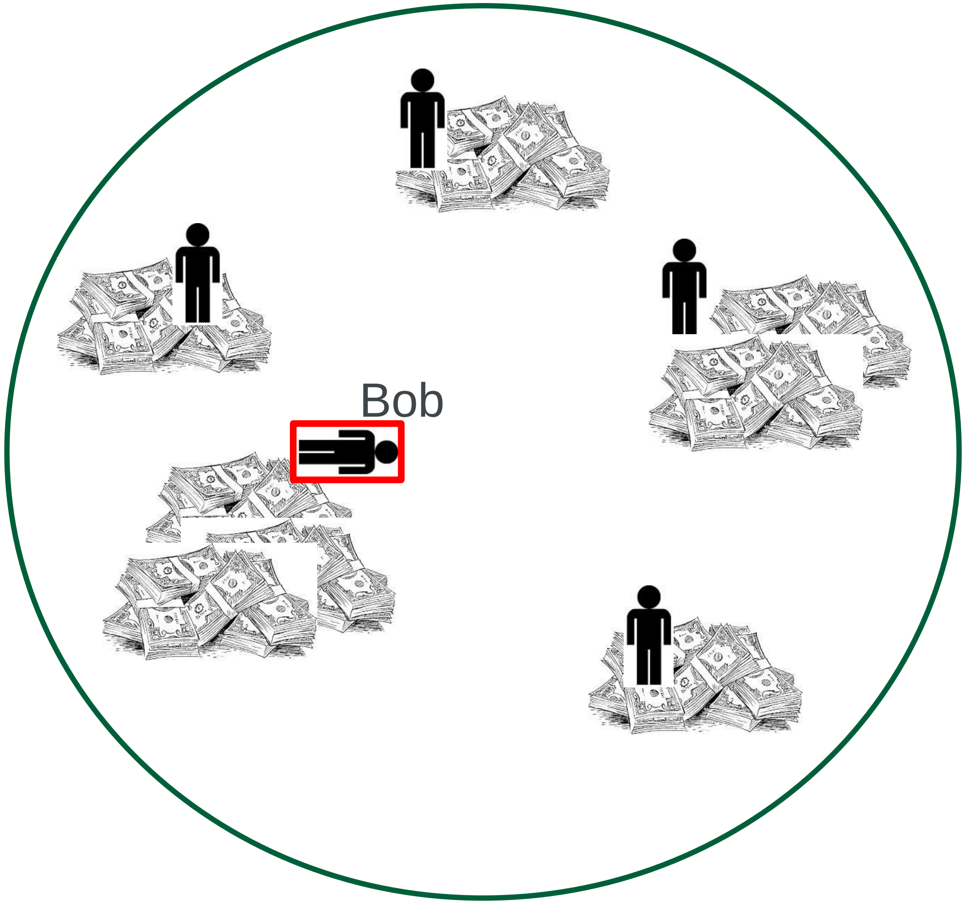
Longevity risk pooling



Share out remaining funds of Bob.

Longevity risk pooling rule [DGN]

- $\lambda^{(i)}$ = Force of mortality of i^{th} member at time T.

- $W^{(i)}$ = Fund value of i^{th} member at time T.

- Payment (longevity credit) to i^{th} member:

$$\frac{\lambda^{(i)} \times W^{(i)}}{\sum_{k \in Group} \lambda^{(k)} \times W^{(k)}} \times \{\text{Bob's remaining fund value}\}$$

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Example I(i): A dies

Member	Force of mortality	Fund value before A dies	Force of mortality x Fund value	Longevity credit from A's fund value = 100 x (4)/Sum of (4)	Fund value afer A dies
(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**
A	0.01	100	1	10	10 = 100-100+10
B	0.01	200	2	20	220 = 200+20
C	0.01	300	3	30	330 = 300+30
D	0.01	400	4	40	440 = 400+40
Total		**1000**	**10**	**100**	**1000**

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Example I(ii): D dies

Member	Force of mortality	Fund value before D dies	Force of mortality x Fund value	Longevity credit from D's fund value = 400 x (4)/Sum of (4)	Fund value afer D dies
(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**
A	0.01	100	1	40	140 = 100+40
B	0.01	200	2	80	280 = 200+80
C	0.01	300	3	120	420 = 300+120
D	0.01	400	4	160	160 = 400-400+160
Total		**1000**	**10**	**400**	**1000**

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Example 2(i): A dies

Member	Force of mortality	Fund value before A dies	Force of mortality x Fund value	Longevity credit from A's fund value = 100 x (4)/Sum of (4)	Fund value afer A dies
(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**
A	0.04	100	4	20	20 = 100-100+20
B	0.03	200	6	30	230 = 200+30
C	0.02	300	6	30	330 = 300+30
D	0.01	400	4	20	420 = 400+20
Total		**1000**	**20**	**100**	**1000**

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Longevity risk pooling rule

- $q^{(i)}$ = Probability of death of i^{th} member from time T to T+1.

- Unit time period could be 1/12 year, 1/4 year, 1/2 year,…

- Longevity credit paid to i^{th} member:

$$\frac{q^{(i)} \times W^{(i)}}{\sum_{k \in Group} q^{(k)} \times W^{(k)}} \times \{\text{Total fund value of members dying between time } T \text{ and } T+1\}$$

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Example 3: larger group, total assets of group €85,461,500.

Age *x* of member	Prob. of death from age *x* to *x*+1	Fund value of each member	Number of members at age *x*		
(1)	**(2)**	**(3)**	**(4)**		
75	0.035378	€100,000	100		
76	0.039732	€96,500	96		
77	0.044589	€93,000	92		
78	0.049992	€89,500	88		
:	:	:	:		
100	0.36992	€12,500	1		
Total	(S1MPA)		**1,121**		

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Example 3: larger group, total assets of group €85,461,500.

Age *x* of member	Prob. of death from age *x* to *x*+1	Fund value of each member	Number of members at age *x*	Prob. of death multiplied by Fund value = (2)x(3)	Per member, share of funds of deceased members = (5)/sum of (4)x(5)
(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**
75	0.035378	€100,000	100	3,537.80	0.00056
76	0.039732	€96,500	96	3,834.14	0.00060
77	0.044589	€93,000	92	4,146.78	0.00065
78	0.049992	€89,500	88	4,474.28	0.00070
:	:	:	:	:	:
100	0.36992	€12,500	1	4,624.00	0.00073
Total	(S1MPA)		**1,121**		

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Example 3: larger group, total assets of group €85,461,500.

Age *x* of member	Prob. of death from age *x* to *x*+1	Fund value of each member	Number of members at age *x*	Observed number of deaths from age *x* to *x*+1	Total funds released by deaths = (3)x(7)
(1)	**(2)**	**(3)**	**(4)**	**(7)**	**(8)**
75	0.035378	€100,000	100	2	€200,000
76	0.039732	€96,500	96	2	€193,000
77	0.044589	€93,000	92	0	€0
78	0.049992	€89,500	88	5	€447,500
:	:	:	:	:	:
100	0.36992	€12,500	1	0	€0
Total	(S1MPA)		**1,121**	**97**	**€5,818,500**

Example 3: larger group, total assets of group €85,461,500.

					Total funds released by deaths = (3)x(7)
					(8)
					5,818,500

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Example 3: larger group, total assets of group €85,461,500.

5 818 500

					Total funds released by deaths = (3)x(7)
					(8)
					€5,818,500

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Example 3: larger group, total assets of group €85,461,500.

5 818 500

Age *x* of member	Prob. of death from age *x* to *x*+1	Fund value of each member	Number of members at age	Prob. of death times Fund value = (2)x(3)	Per member, share of funds of deceased members = (5)/sum of (4)x(5)
(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**
75	0.035378	€100,000	100	3,537.80	0.00056
76	0.039732	€96,500	96	3,834.14	0.00060
77	0.044589	€93,000	92	4,146.78	0.00065
78	0.049992	€89,500	88	4,474.28	0.00070
:	:	:	:	:	:
100	0.36992	€12,500	1	4,624.00	0.00073
Total	(S1MPA)		**1,121**		

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Example 3: larger group, total assets of group €85,461,500.

5 818 500

Age *x* of member	Prob. of death from age *x* to *x*+1	Fund value of each member	Number of members at age	Prob. of death times Fund value = (2)x(3)	Longevity credit per member = (6) x sum of (8)
(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(9)**
75	0.035378	€100,000	100	3,537.80	€3,237.33
76	0.039732	€96,500	96	3,834.14	€3,508.50
77	0.044589	€93,000	92	4,146.78	€3,794.58
78	0.049992	€89,500	88	4,474.28	€4,094.28
:	:	:	:	:	:
100	0.36992	€12,500	1	4,624.00	€4,231.28
Total	(S1MPA)		**1,121**		

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Example 3: larger group, total assets of group €85,461,500.

Age *x* of member	Prob. of death from age *x* to *x*+1	Fund value of each member	Longevity credit per member = (6) x sum of (8)	Fund value of survivor at age *x*+1	Fund value of deceased at age *x*+1
(1)	**(2)**	**(3)**	**(9)**	**(10)**	**(11)**
75	0.035378	€100,000	€3,237.33	€103,237.33	€3,237.33
76	0.039732	€96,500	€3,508.50	€100,008.50	N/A
77	0.044589	€93,000	€3,794.58	€96,794.58	€3,794.58
78	0.049992	€89,500	€4,094.28	€93,594.28	€4,094.28
:	:	:	:	:	:
100	0.36992	€12,500	€4,231.28	€16,731.28	N/A

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Longevity risk pooling [DGN] - features

- Total asset value of group is unchanged by pooling.
  - Individual values are re-arranged between the members
- Expected actuarial gain = 0, for all members at all times.
  - Actuarial gain of member (x) from time T to $T+1$

=

\+ Longevity credits gained by (x) from deaths (including (x)'s own death) between time T and $T+1$

\- Loss of (x)'s fund value if (x) dies between times T and $T+1$.

i.e. the pool is actuarially fair at all times: no-one *expects* to gain from pooling.

Longevity risk pooling [DGN] - features

- Expected longevity credit =

$$\{Prob\ of\ death\ of\ (x)\} \times \{Fund\ value\ of\ (x)\}$$

$$\times \left(1 - \frac{\{Prob\ of\ death\ of\ (x)\} \times \{Fund\ value\ of\ (x)\}}{\sum_{y \in Group} \{Prob\ of\ death\ of\ (y)\} \times \{Fund\ value\ of\ (y)\}}\right).$$

- Expected longevity credit tends to

$$\{Prob\ of\ death\ of\ (x)\} \times\ \{Fund\ value\ of\ (x)\}$$

as group gets bigger.

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Longevity risk pooling [DGN] - features

- There will always be some volatility in the longevity credit:
  - Actual value ≠ expected value (no guarantees)
  - But longevity credit ≥ 0, i.e. never negative.
  - Loss occurs only upon death.

- Volatility in longevity credit can replace investment return volatility.

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Longevity risk pooling [DGN] - features

- Scheme works for any group:
  - Actuarial fairness holds for any group composition, but
  - Requires a payment to estate of recently deceased.
  - Sabin [see Part IV] proposes a survivor-only payment. However, it requires restrictions on membership.
  - Should it matter? Not if group is well-diversified (Law of Large Numbers holds) – then schemes should be equivalent.

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Longevity risk pooling [DGN] - features

- Increase expected lifetime income
- Reduce risk of running out of money before death
- Non-negative return, except on death
- Update force of mortality, periodically.

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Longevity risk pooling [DGN] - features

- ``Cost'' is paid upon death, not upfront like life annuity.
- Mitigates longevity risk, but does not eliminate it.
- Anti-selection risk remains, as for life annuity. Waiting period?

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Longevity risk pooling [DGN] - features

- Splits investment return from longevity credit to enable:
  - Fee transparency,
  - Product innovation.

Longevity risk pooling [DGN] –analysis

- Compare:
  a) Longevity risk pooling, versus
  b) Equity-linked life annuity, paying actuarial return ($\lambda^{(i)}$ – Fees) x $W^{(i)}$.

Fees have to be <0.5% for b) to have higher expected return in a moderately-sized (600 members), heterogeneous group [DGN].

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Longevity risk pooling [DGN] – some ideas

- Insurer removes some of the longevity credit volatility, e.g. guarantees a minimum payment for a fee [DY].

- Allow house as an asset – monetize without having to sell it before death [DY].

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Longevity risk pooling [DGN] – some ideas

- Pay out a regular income with the features:
  - Each customer has a ring-fenced fund value.
  - Explicitly show investment returns and longevity credits on annual statements.
  - Long waiting period before customer's assets are pooled, to reduce adverse selection risk, e.g. 10 years.
  - More income flexibility.
  - Opportunity to withdraw a lumpsum from asset value.
  - Update forces of mortality periodically.

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II. One way of pooling longevity risk - Summary

- DGN method of pooling longevity risk
  - Explicit scheme.
  - Everything can be different: member characteristics, investment strategy.
- Can provide a higher income in retirement.
- Reduces chance of running out of money in retirement.
- May also result in a higher bequest.
- Transparency may encourage more people to “annuitize”.

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Overview of entire session

I. Motivation

II. One way of pooling longevity risk

III. Classification of methods & discussion

IV. A second explicit scheme

V. An implicit scheme

VI. Summary and discussion

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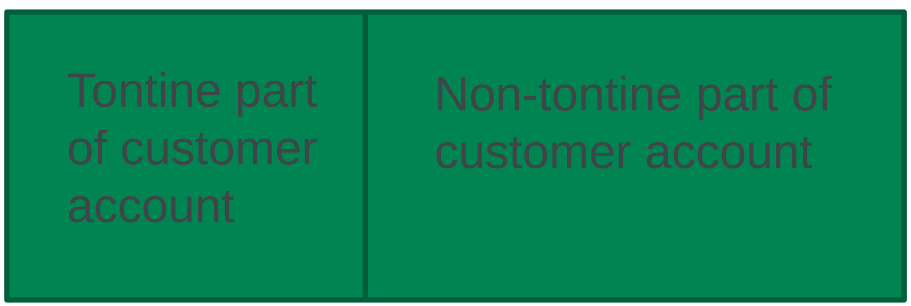
Classification of methods

- Explicit tontines: e.g. [DGN] (Part II) and Sabin (Part IV)
- Individual customer accounts
- Customer chooses investment strategy
- Customer chooses how much to allocate to tontine
- Initially:



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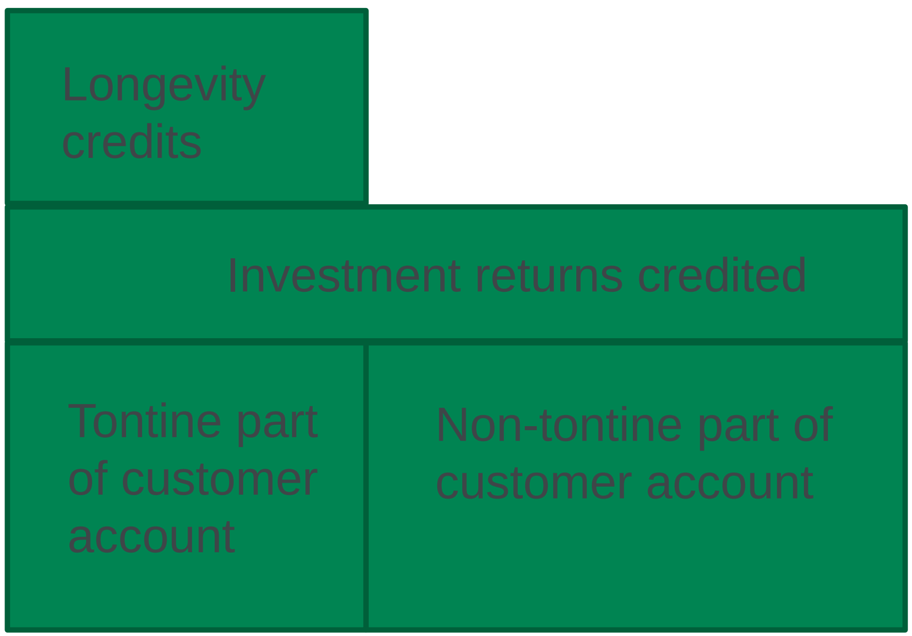
Explicit tontines

- Add in returns and credits:

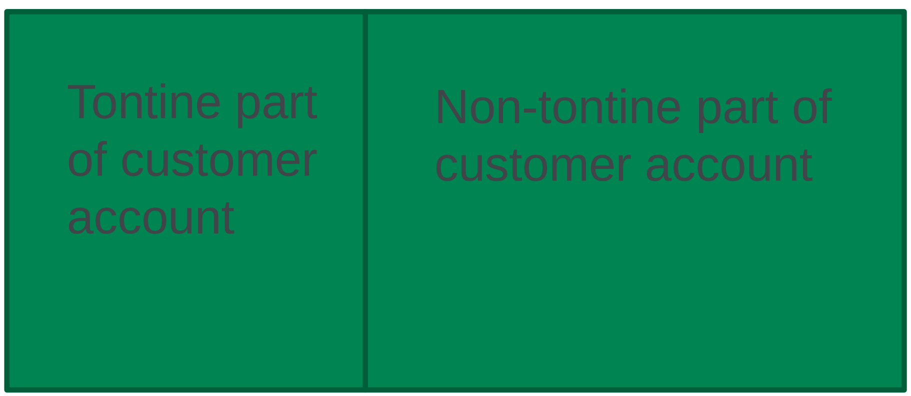


Explicit tontines

- Subtract income withdrawn by customer: chosen by customer, subject to limitations (avoid anti-selection/moral hazard)

Explicit tontines

- Either re-balance customer account to maintain constant percentage in tontine, or
- Keep track of money in and out of each sub-account



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Implicit tontines

- Implicit tontines: e.g. GSA (Part V)
- Works like a life annuity
- Likely to assume that idiosyncratic longevity risk is zero
- Customers are promised an income in exchange for upfront payment
- Income adjusted for investment and mortality experience
- The explicit tontines can be operated as implicit tontines

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Implicit methods

- Same investment strategy for all customers
- Less clear how to allow flexible withdrawals (e.g. GSA not actuarially fair except for perfect pool)
- Might be easier to implement from a legal/regulatory viewpoint

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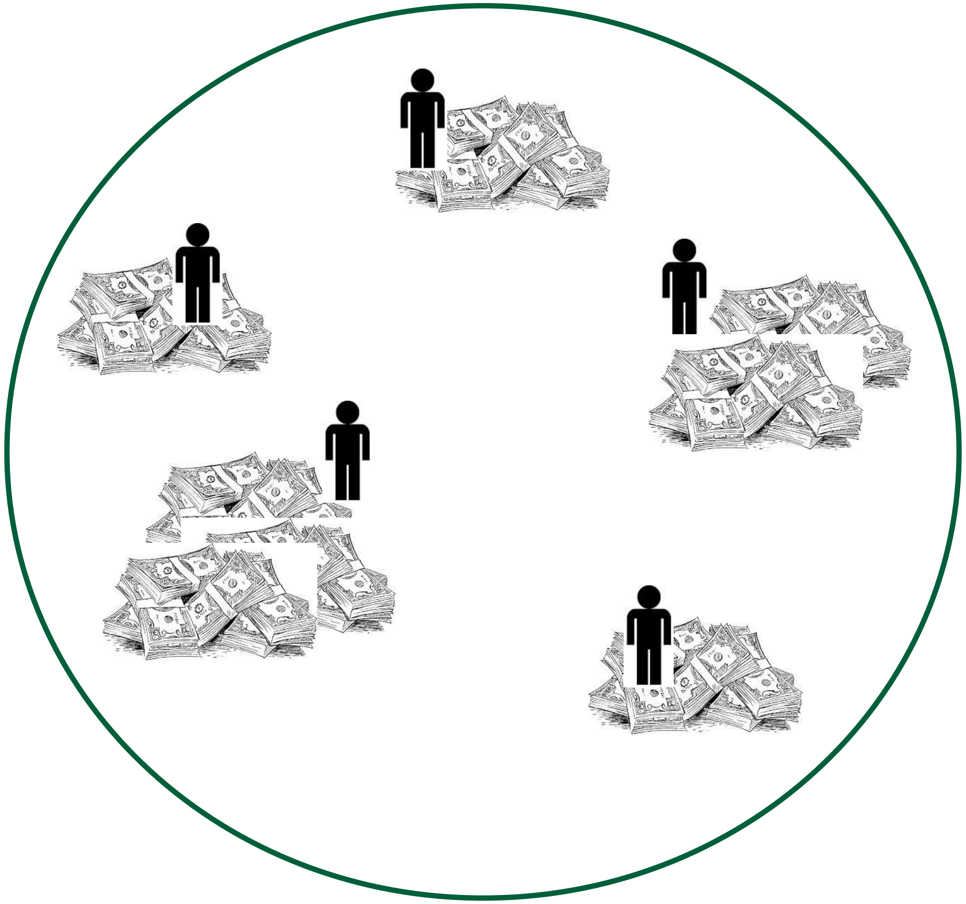
A second explicit scheme [Sabin] - overview

- [DGN] scheme works for any heterogeneous group.
- Simple rule for calculating longevity credits.
- Requires payment to the estate of recently deceased to be actuarially fair.
- [Sabin] shares out deceased’s wealth only among the survivors.
- Restrictions on the group composition to maintain actuarial fairness.
- Longevity credit allocation in [Sabin] is more complicated.

Longevity risk pooling [Sabin]



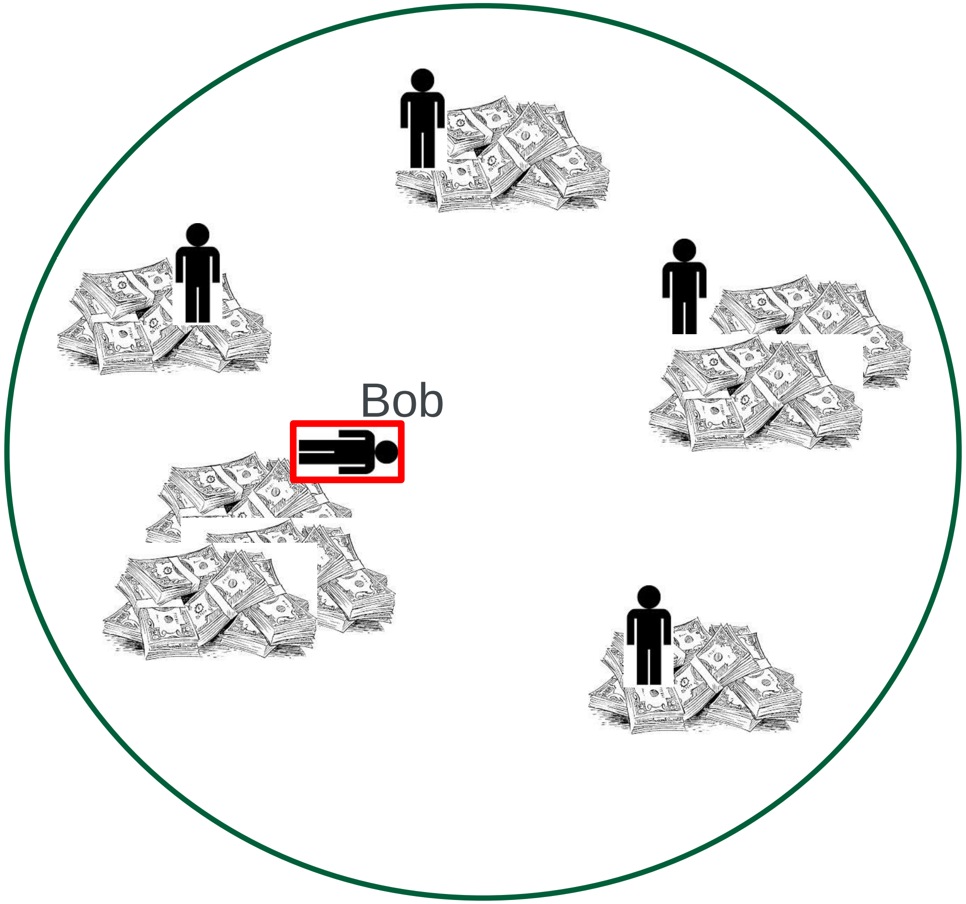
Pool risk over lifetime

Individuals make their own investment decisions

Individuals withdraw income from their own funds

However, when someone dies at time *T*…

Longevity risk pooling [Sabin]



Share out remaining funds of Bob.

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Longevity risk pooling rule [Sabin]

- Longevity credit paid to i^{th} member is

$$\alpha_{i,Bob} \times \{\text{Bob's remaining fund value}\},$$

- $\alpha_{i,Bob}$ = Share of Bob's fund value received by i^{th} member, with $\alpha_{i,Bob} \in [0,1]$.

- Payment to survivors only, so $\alpha_{Bob,Bob} = -1$.
- No more and no less than Bob's fund is shared out, so

$$\sum_{i \neq Bob} \alpha_{i,Bob} = 1.$$

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Longevity risk pooling rule [Sabin]

- Impose actuarial fairness: Expected gain from tontine is zero.
- $\alpha_{i,d}$ = Share of deceased *d*'s fund value received by *i*th member.
- $\lambda^{(i)}$ = Force of mortality of *i*th member at time *T*.
- $W^{(i)}$ = Fund value of *i*th member at time *T.*
- Expected gain of *i*th member from tontine is

$$\sum_{d \neq i} \lambda_d \alpha_{i,d} W_d - \lambda_i W_i = 0.$$

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Longevity risk pooling rule [Sabin]

Simple setting of 3 members.

Then we must solve for $\left(\alpha_{i,j}\right)_{i,j=1,2,3}$ the system of equations

$$
\begin{aligned}
&\lambda_2\alpha_{12}W_2 + \lambda_3\alpha_{13}W_3 - \lambda_1 W_1 = 0\\
\lambda_1\alpha_{21}W_1 &+ \lambda_3\alpha_{23}W_3 - \lambda_2 W_2 = 0\\
\lambda_1\alpha_{31}W_1 + \lambda_2\alpha_{32}W_2 &- \lambda_3 W_3 = 0
\end{aligned}
$$

subject to the constraints

$$\sum_{i\neq j}\alpha_{ij} = 1, \text{for } j = 1,2,3,$$

$\alpha_{ij} \in [0,1]$ for all $i \neq j$.

Longevity risk pooling rule [Sabin]

- Does a solution exist? [Sabin] proves that for each member *i* in the group,

$$\sum_{k \in group} \lambda_k W_k \geq 2\lambda_i W_i$$

is a necessary and sufficient condition for $\left(\alpha_{i,j}\right)_{i,j \in Group}$ to exist.

- In general, there is no unique solution.
- [Sabin] and [Sabin2011b] contain algorithms to solve the system of equations.

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Example 4(i): [Sabin, Example 1] A dies

Member i	$\lambda_i / \sum_{k \in \{A,B,C,D\}} \lambda_k$	Fund value before A dies	$\alpha_{i,A}$	Longevity credit from A's fund value = $\alpha_{i,A}$ x 2	Fund value afer A dies = (3) + (5)
(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**
A	0.55464	2	-1	-2	0
B	0.15983	6	0.61302	1.22604	7.22604
C	0.14447	3	0.23766	0.47532	3.47532
D	0.14107	2	0.14932	0.29864	2.29864
Total	1.0000	**13**	**0.00000**	**0.00000**	**13.00000**

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Example 4(ii): [Sabin, Example 1] B dies

Member i	$\lambda_i / \sum_{k\in\{A,B,C,D\}} \lambda_k$	Fund value before B dies	$\alpha_{i,B}$	Longevity credit from B's fund value = $\alpha_{i,B}$ x 6	Fund value afer B dies = (3) + (5)
(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**
A	0.55464	2	0.75754	4.54524	6.54524
B	0.15983	6	-1	-6	0
C	0.14447	3	0.14814	0.88884	3.88884
D	0.14107	2	0.09432	0.56592	2.56592
Total	1.0000	**13**	**0.00000**	**0.00000**	**13.00000**

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Example 5(i): A dies – one solution

Member	Force of mortality	Fund value before A dies	$\alpha_{i,A}$	Longevity credit from A's fund value = $\alpha_{i,B}$ x 150	Fund value afer A dies
(1)	(2)	(3)	(4)	(5)	(6)
A	0.04	150	-1	-150	0
B	0.03	200	1/3	50	250
C	0.02	300	1/3	50	350
D	0.01	600	1/3	50	650
Total		**1250**	**0**	**0**	**1250**

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Example 5(i): Full solution

Member	$\alpha_{i,A}$	$\alpha_{i,B}$	$\alpha_{i,C}$	$\alpha_{i,D}$
(1)	**(2)**	**(3)**	**(4)**	**(5)**
A	-1	1/3	1/3	1/3
B	1/3	-1	1/3	1/3
C	1/3	1/3	-1	1/3
D	1/3	1/3	1/3	-1
Total	**0**	**0**	**0**	**0**

Example 5(ii): A dies – another solution (not so nice)

Member	Force of mortality	Fund value before A dies	$\alpha_{i,A}$	Longevity credit from A's fund value = $\alpha_{i,B}$ x 150	Fund value afer A dies
(1)	**(2)**	**(3)**	**(4)**	**(5)**	**(6)**
A	0.04	150	-1	-150	0
B	0.03	200	0	0	200
C	0.02	300	0	0	300
D	0.01	600	1	150	750
Total		**1250**	**0**	**0**	**1250**

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Example 5(ii): Full solution

Member	$\alpha_{i,A}$	$\alpha_{i,B}$	$\alpha_{i,C}$	$\alpha_{i,D}$
(1)	**(2)**	**(3)**	**(4)**	**(5)**
A	-1	0	0	1
B	0	-1	1	0
C	0	1	-1	0
D	1	0	0	-1
Total	**0**	**0**	**0**	**0**

Choosing a solution [Sabin]

- [Sabin] suggests minimizing the variance of $(\alpha_{i,j})$, among other possibilities. However, for *M* group members, the algorithm has run-time $\mathcal{O}(M^3)$.
- He suggests another approach (called *Separable Fair Transfer Plan*) which has run-time $\mathcal{O}(M)$.

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A second explicit scheme [Sabin] - summary

- Shares out deceased’s wealth only among the survivors.
- Restrictions on the group composition to maintain actuarial fairness.
- Longevity credit allocation is more complicated.
- No unique solution, but a desired solution can be chosen.
- For implementation, [Sabin] can operate like [DGN].

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An implicit scheme [GSA] – Group Self-Annuitisation

- Group Self-Annuitisation (GSA) pays out an income to its members.
- Collective fund, one investment strategy.
- Income is adjusted for mortality and investment experience.
- Income calculation assumes Law of Large Numbers holds.
- Works for heterogeneous membership.
- But assume homogeneous example next.

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[GSA] – Homogeneous membership

- Group of M homogeneous members, all age 65 initially
- Track total fund value F_n.
- Each receives a payment at start of first year

$$B_0 = \frac{1}{M}\frac{F_0}{\ddot{a}_{65}} = \frac{1}{l^*_{65}}\frac{F_0}{\ddot{a}_{65}},$$

with $l^*_{65} = M$ (actual number alive at age 65)

and

$$\ddot{a}_{65} = 1 + \sum_{k=1}^{\infty}(1+R)^{-k} \times {}_k p_{65}.$$

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[GSA] – Homogeneous membership

- End of first year, total fund value in GSA is

$$F_1 = (F_0 - l_0^* B_0) \times (1 + R),$$

where R is the actual investment return in the first year (assume it equals its expected return R).

- l_{66}^* members alive (expected number was $l_{65}^* \times p_{65}$.).
- Each survivor receives a payment at start of second year

$$B_1 = \frac{1}{l_{66}^*} \frac{F_1}{\ddot{a}_{66}},$$

$$\ddot{a}_{66} = 1 + \sum_{k=1}^{\infty} (1 + R)^{-k} \times_k p_{66}.$$

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[GSA] – Homogeneous membership

- Straightforward to show

$$B_1 = B_0 \times \frac{p_{65}}{p^*_{65}},$$

where

- p^*_{65} is the empirical probability of one-year survival, and
- p_{65} is the estimated probability of one-year survival.

- More generally,

$$B_n = B_{n-1} \times \frac{p_{65+n-1}}{p^*_{65+n-1}}.$$

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[GSA] – Homogeneous membership

- Allow for actual annual investment returns $R_1^*, R_2^*, \ldots$ in year 1,2,…
- Then end of first year, total fund value in GSA is

$$F_1 = (F_0 - l_0^* B_0) \times (1 + R_1^*).$$

- Benefit paid to each survivor at start of second year is

$$B_1 = B_0 \times \frac{p_{65}}{p_{65}^*} \times \frac{1 + R_1^*}{1 + R}.$$

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[GSA] – Homogeneous membership

- More generally,

$$B_n = B_{n-1} \times \frac{p_{65+n-1}}{p^*_{65+n-1}} \times \frac{1 + R^*_n}{1 + R}$$

- Or

$$B_n = B_{n-1} \times MEA_n \times IRA_n,$$

where

MEA_n= Mortality Experience Adjustment

IRA_n=Interest Rate Adjustment

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[GSA] – Different initial contributions

- Group of M members, all age 65 initially
- Member *i* pays in amount $F_0^{(i)}$.
- Total fund value $F_0 = \sum_{i=1}^{M} F_0^{(i)}$.
- Member *i* receives a payment at start of first year

$$B_0^{(i)} = \frac{F_0^{(i)}}{\ddot{a}_{65}}$$

with $\ddot{a}_{65} = 1 + \sum_{k=1}^{\infty} (1+R)^{-k} \times {}_k p_{65}$.

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[GSA] – Different initial contributions

- At end of first year, fund value of member *i* is

$$F_1 = \left(F_0 - \sum_{i=1}^{M} B_0^{(i)}\right) \times (1 + R_1^*),$$

where R_1^* is the actual investment return in the first year.

- Fund value of member *i* is

$$F_1^{(i)} = \left(F_0^{(i)} - B_0^{(i)}\right) \times (1 + R_1^*).$$

- Fund value of members dying over first year is distributed among survivors in proportion to fund values.

[GSA] – Different initial contributions

- If member *i* is alive at start of second year, they get a benefit payment

$$B_1^{(i)} = \frac{1}{\ddot{a}_{66}}\left(F_1^{(i)} + \frac{F_1^{(i)}}{\sum_{s\in Survivors} F_1^{(s)}} \times \sum_{d\in Dead} F_1^{(d)}\right).$$

- Can show that

$$B_1^{(i)} = B_0^{(i)} \times \frac{p_{65}}{\sum_{s\in Survivors} F_1^{(s)}/F_1} \times \frac{1+R_1^*}{1+R}$$

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[GSA] – Different initial contributions

- More generally,

$$B_n = B_{n-1} \times \frac{p_{65+n-1}}{\sum_{s \in Survivors} F_n^{(s)} / F_n} \times \frac{1+R_n^*}{1+R},$$

- which has the form

$$B_n = B_{n-1} \times MEA_n \times IRA_n,$$

where MEA_n= Mortality Experience Adjustment and IRA_n=Interest Rate Adjustment.

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[GSA] – Different initial contributions

- [GSA] extend to members of different ages.

- Further allow for updates to future mortality,

$$B_n = B_{n-1} \times MEA_n \times IRA_n \times CEA_n,$$

where CEA_n= Changed Expectation Adjustment = $\frac{\ddot{a}^{\text{old}}_{65+n-1}}{\ddot{a}^{\text{new}}_{65+n-1}}$.

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[GSA] – analysis

- Same investment strategy for all members: strategy for 65 year old = strategy for 80 year old? Are all 65 year olds the same?
- Fixed benefit calculation - no choice.
- Not actuarially fair: $F_0^{(i)} \neq \mathbb{E}(\text{Discounted future benefits})$.
- Two finite groups with different wealth, otherwise identical.
  - Higher wealth group lose: $F_0^{(i)} > \mathbb{E}(\text{Discounted future benefits})$
  - Higher wealth group expect higher benefits if groups had same wealth.
  - Only significant in small or highly heterogeneous groups.

[Donnelly2015]

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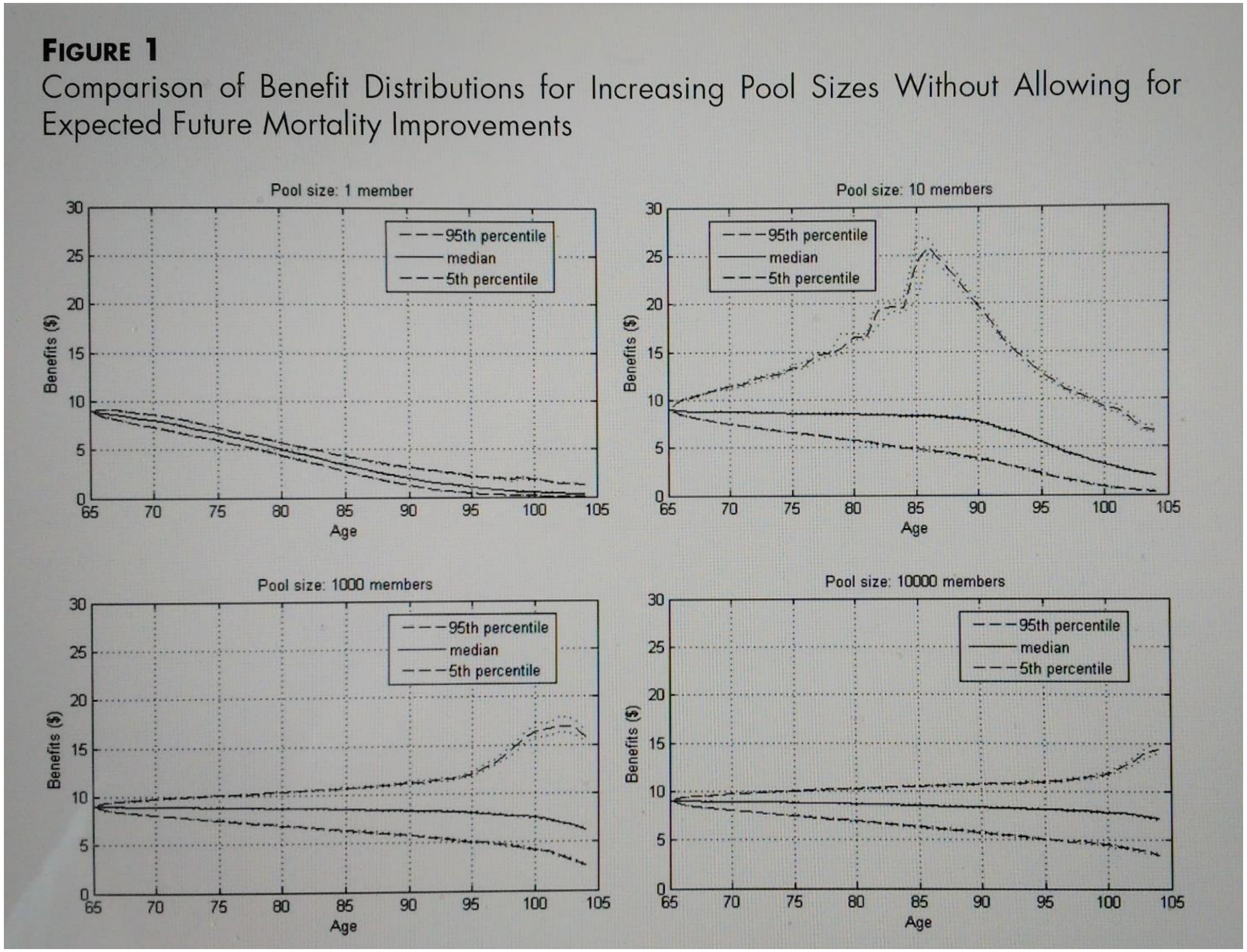
GSA – analysis [QiaoSherris], Figure 1



- \$100 paid on entry at age 65.
- Max age 105.
- Single cohort.
- Interest rate 5% p.a.
- Allow for systemic mortality changes

$$\mu_{x,t} = Y_t^{(1)} + Y_t^{(2)} 1.0966^x,$$
$$dY_t^{(1)} = a_1 dt + \sigma_1 dW_t^{(1)},$$
$$dY_t^{(2)} = a_2 dt + \sigma_2 dW_t^{(2)},$$
$$d\left[W_t^{(1)}, W_t^{(2)}\right] = 0.929 dt$$

with $\mu_{x,t} := 0 \; if \, \mu_{x,t} < 0$.

- Don't allow for future expected improvements in annuity factor.

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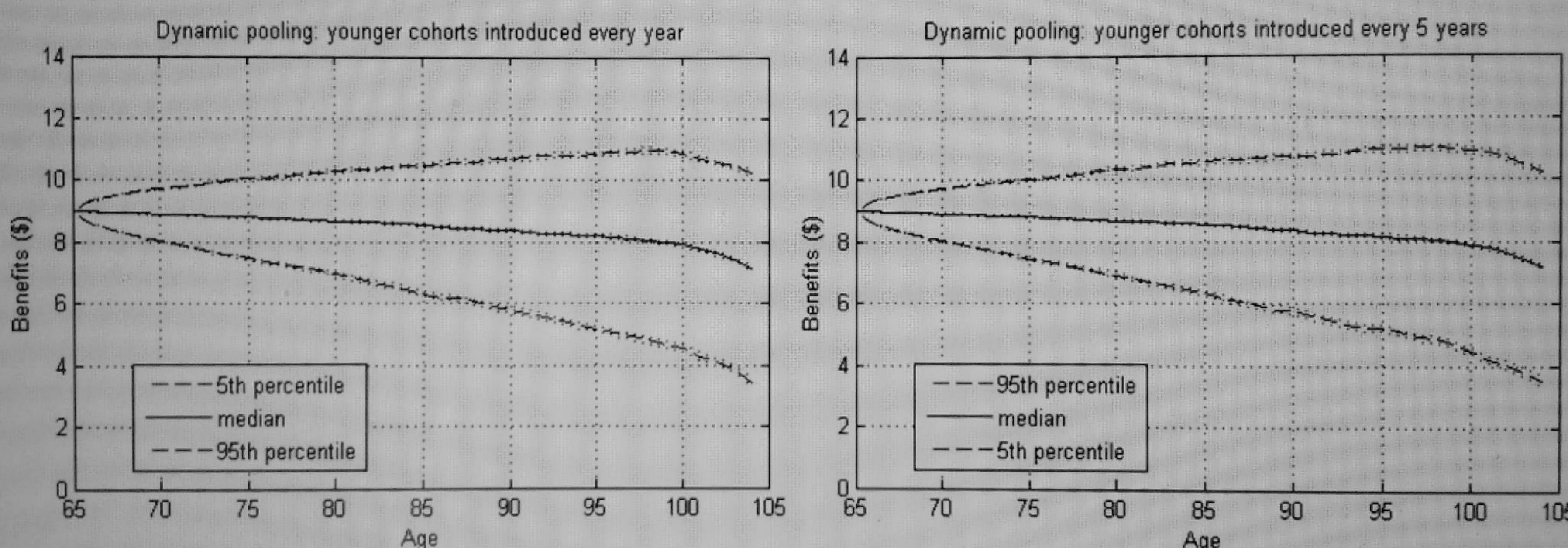
GSA – analysis [QiaoSherris], Figure 2

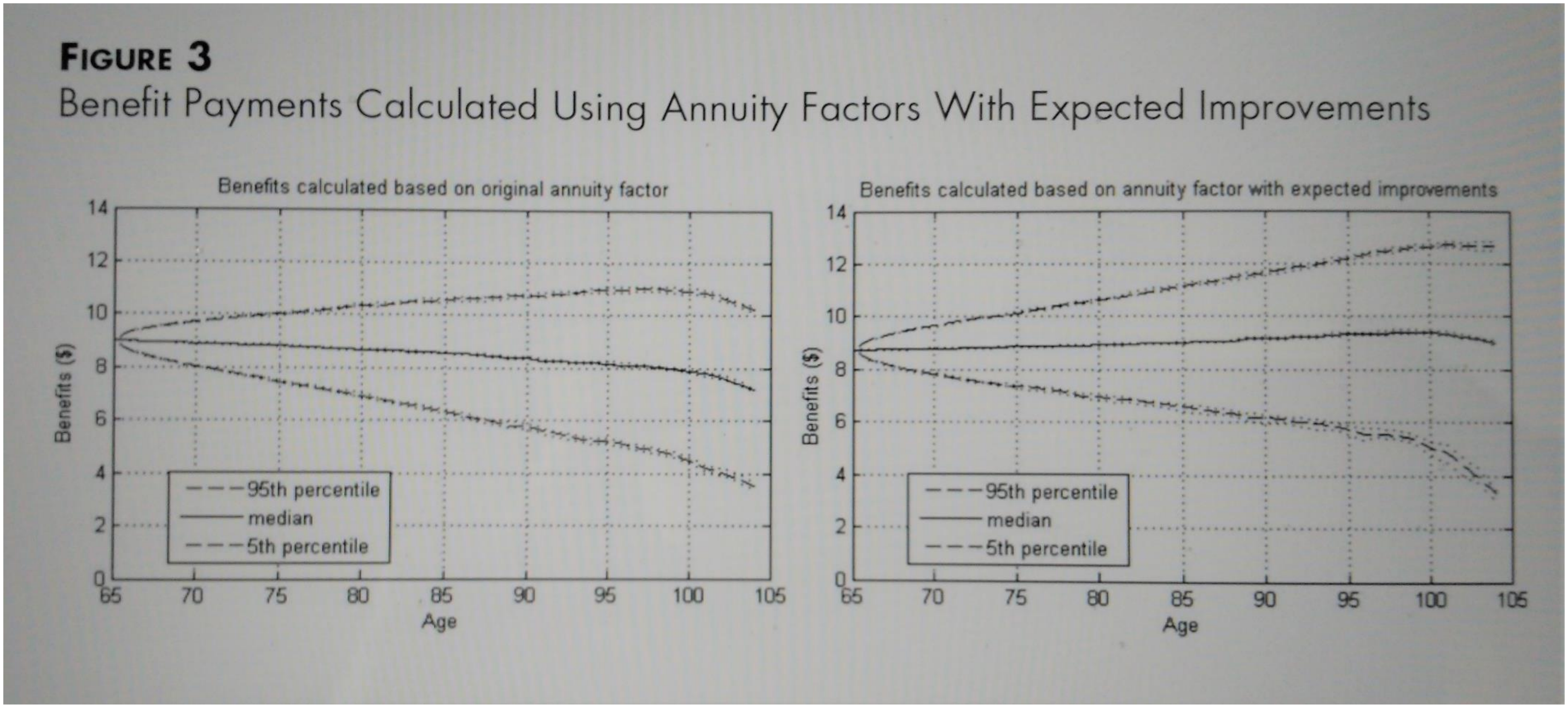
FIGURE 2

Benefit Distributions Dynamic Pooling Every Year (1,000 New 65-Year-Olds Entering Every Year and Every 5 Years)



GSA – analysis [QiaoSherris], Figure 3

- 1000 members age 65 join every 5 years.
- Update annuity factor to allow for mortality improvements.



Group Self-Annuitisation - Summary

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- Income is adjusted for mortality and investment experience.
- Works for heterogeneous membership.

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Summary and discussion

- Reduce risk of running out of money
- Provide a higher income than living off investment returns alone
- Should be structured to provide a stable, fairly constant income (**not** increasing exponentially with the longevity credit!)
- Two types of tontine:
  - Explicit: Longevity credit payment
  - Implicit: Income implicitly includes longevity credit

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Summary and discussion

- Looked at two actuarially fair explicit tontines [DGN], [Sabin].
- Enable tailored solution: e.g. individual investment strategy.
- Easier to add product innovation: e.g. partial guarantees.
- Others have been proposed, not necessarily actuarially fair.
- In practice, Mercer Australia LifetimePlus appears to be an explicit tontine (though income profile unattractive).
- [GSA] is an implicit tontine.
- Isn't actuarially fair, but shouldn't matter if enough members.
- In practice, TIAA-CREF annuities are similar.

Questions

Comments

The views expressed in this presentation are those of the presenter.

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The '**Minimising Longevity and Investment Risk while Optimising Future Pension Plans**' research programme is being funded by the ARC.

www.actuaries.org.uk/arc

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