On Contemporary Mortality Models for Actuarial Use II: Principles

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Contents

QUESTION: WHAT DO WE MEAN BY A CONTINUOUS-TIME MORTALITY MODEL?

1. Discrete time $\Rightarrow$ complicated events!
2. Breaking down events — the Bernoulli 'atom'
3. Building up events — the product integral
4. Data — the stochastic switch $Y(t)$
   - Survival models
   - Pseudo-Poisson models
   - True Poisson models

Models: q-type and μ-type

Forfar, D.O., McCutcheon, J.J. & Wilkie, A.D. (1988). *On Graduation by Mathematical Formula*. Journal of the Institute of Actuaries, **115**, 1–149.

FMW graduated models using estimators of three parameters:

- q_x the one-year probability of death;
- μ_x the hazard rate*; or
- m_x the central rate of mortality.

* ‘force of mortality’ if you prefer

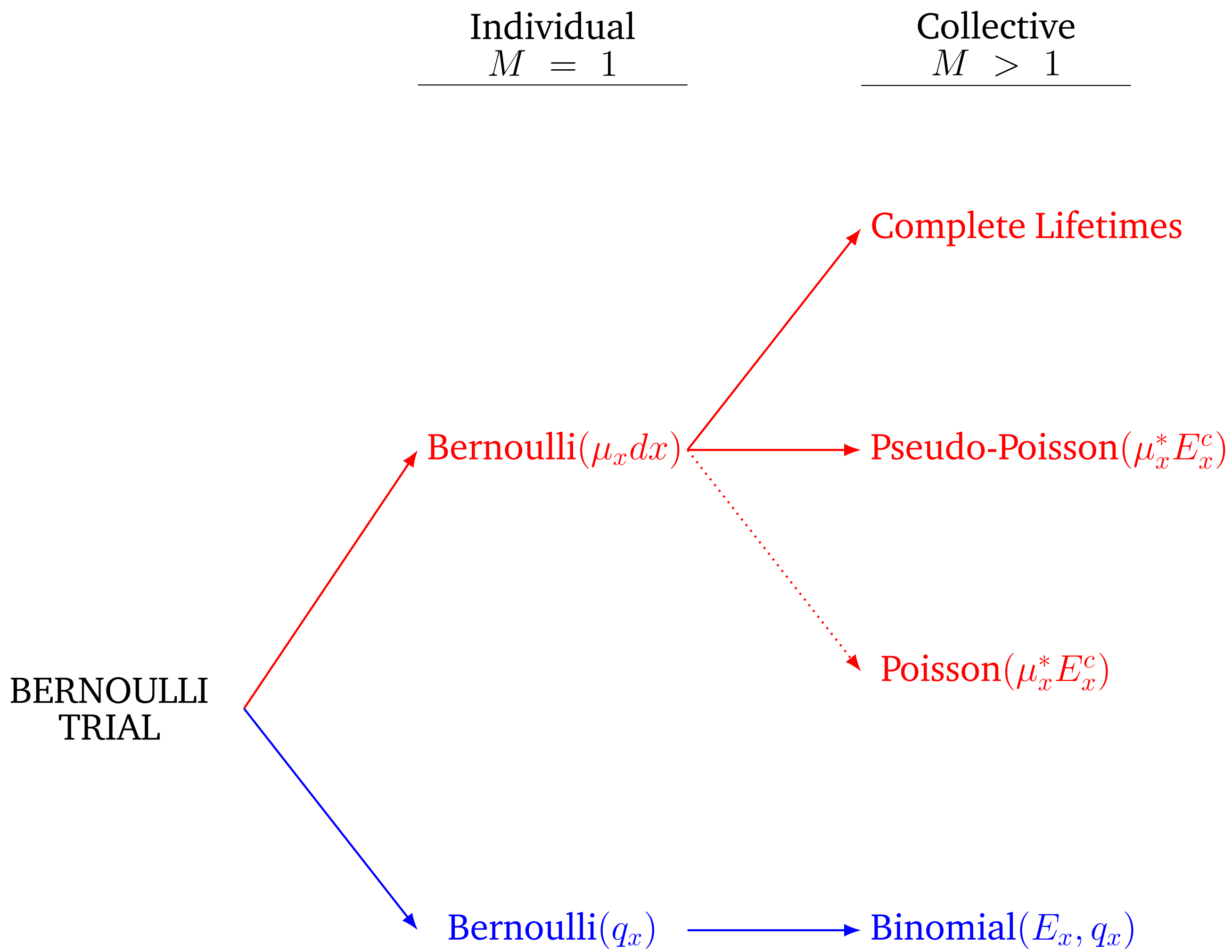
Models: q-type and μ-type

Flaws with simple models.

1. **q-type Model: Binomial Distribution**
   - Assumes E_x persons exposed for a whole year BUT ...
   - ... some will leave before the year-end ...
   - ... while others will join part-way through the year ...
   - ... so we can't have a Binomial distribution.
2. **μ-type Model: Poisson Distribution**
   - Knowing we have M individuals in the study (as we usually do) ...
   - ... the probability of more than M deaths is zero ...
   - ... so we can't have a Poisson distribution.

'Fixing' the Binomial model leads us further into the weeds. Fixing the Poisson model leads to enlightenment!

Bernoulli Family Tree



The Lower Branch: The Binomial/Bernoulli Model

Observe M lives $i = 1, 2, \ldots, M$ for one year, define 'indicator' of death d_i:

$$d_i = \begin{cases} 1 & \text{if life } i \text{ dies} \\ 0 & \text{if life } i \text{ survives} \end{cases}$$

Binomial likelihood is:

$$\begin{aligned} L_i &\propto (1 - q_x)^{M - \sum d_i} \, (q_x)^{\sum d_i} \\ &= \prod_{i=1}^{M} (1 - q_x)^{1 - d_i} \, (q_x)^{d_i}. \end{aligned}$$

... a product of **Bernoulli** likelihoods for each life.

. . . But THIS Bernoulli Model is Still a Complicated Thing!

Define T_x = random lifetime of (x) and consider $p_x = \mathrm{P}[T_x > 1]$:

Event $\{T_x > 1\}$ is highly composite:

$$
\begin{aligned}
p_x &= {}_1p_x \\
&= {}_{0.5}p_x \times {}_{0.5}p_{x+0.5} \\
&= {}_{0.25}p_x \times {}_{0.25}p_{x+0.25} \times {}_{0.5}p_{x+0.5} \\
&= {}_{0.125}p_x \times {}_{0.125}p_{x+0.125} \times {}_{0.25}p_{x+0.25} \times {}_{0.5}p_{x+0.5} \cdots \\
&= \ldots \text{and so on, } \textit{ad infinitum}.
\end{aligned}
$$

(Apologies to Zeno!)

In fact, event $\{T_x > 1\}$ is infinitely composite. Survival happens from moment to moment. And $q_x = \mathrm{P}[T_x \leq 1]$ is worse.

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The Basic 'Atom' — An Infinitesimal Bernoulli Trial

The idea of the hazard rate μ_t is the infinitesimal:

$$\mathrm{P}[t < T \leq t + dt \mid T > t] = \mu_t \, dt + o(dt) \approx \mu_t \, dt.$$

For convenience (re)define the indicator:

$$d_i = \Delta N_i(t) = \begin{cases} 1 \text{ if } t < T_i < t + dt \\ 0 \text{ otherwise} \end{cases}$$

$$\mathrm{P}[\text{Obs. in } dt] \quad = \quad (1 - \mu_t \, dt)^{(1 - \Delta N_i(t))} \, (\mu_t \, dt)^{\Delta N_i(t)} = \text{Bernoulli trial}.$$

We have the infinitesimal Bernoulli trial. Not quite right yet, but let's pursue it . . .

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The Product Integral

Revision: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ or $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = e^{-1}$.

$\longrightarrow$ a link between limits of products (product integrals) and limits of sums ordinary integrals) via the exponential function:

$$\prod_{s\in(0,t]} (1 + f(s)\,ds) = \exp\left(\int_0^t f(s)\,ds\right)$$

Choose $f(s) = -\mu_{x+s}$

$$\longrightarrow \quad \prod_{s\in(0,t]} (1 - \mu_{x+s}\,ds) = \exp\left(-\int_0^t \mu_{x+s}\,ds\right) = {}_tp_x.$$

The Product Integral

Let $(a_i, b_i]$ be the time interval under observation by life i. Then:

$$\begin{aligned}
\mathrm{P}[\text{Observation}_i] &= \underbrace{\prod_{(a_i,b_i]} \underbrace{(1-\mu_t\,dt)^{(1-\Delta N_i(t))}(\mu_t\,dt)^{\Delta N_i(t)}}_{\text{Bernoulli 'Atoms'}}}_{\text{Product Integral}} \\
&= \underbrace{\left(\prod_{(a_i,b_i]} (1-\mu_t\,dt)^{(1-\Delta N_i(t))}\right)}_{\text{Contributions from survival}} \times \underbrace{(\mu_{b_i}\,dt)^{\Delta N_i(b_i)}}_{\text{Death/Censoring}} \\
&= \exp\left(-\int_{a_i}^{b_i} \mu_t\,dt\right)(\mu_{b_i}\,dt)^{\Delta N_i(b_i)}.
\end{aligned}$$

Contents

Data: The Stochastic Switch $Y(t)$

Define the process $Y^i(t)$:

$$Y^i(t) = \begin{cases} 1 & \text{if alive and under observation at time } t^- \\ 0 & \text{otherwise} \end{cases}$$

$Y^i(t)$ acts as a stochastic 'switch' depending on the status of (x).

For example, $Y^i(t)\,\mu_t$ is a stochastic hazard rate.

$$Y^i(t)\,\mu_t = \begin{cases} \mu_t & \text{if alive and under observation at time } t^- \\ 0 & \text{otherwise} \end{cases}$$

Data: The Product Integral Likelihood

$Y^i(t) = I_{\{\text{Life } i \text{ alive } \textbf{and under observation}\}}$.

$$(1 - Y^i(t)\,\mu_t\,dt)^{1-\Delta N_i(t)}(Y^i(t)\,\mu_t\,dt)^{\Delta N_i(t)}$$

MICRO: The Bernoulli 'atom' of all Poisson-type likelihoods:

$$\begin{aligned} L_i = \text{P}[\text{Observation}_i] &= \prod_{[0,\infty)} (1 - Y^i(t)\,\mu_t\,dt)^{(1-\Delta N_i(t))}(Y^i(t)\,\mu_t\,dt)^{\Delta N_i(t)} \\ &= \underbrace{\exp\left(-\int_0^\infty Y^i(t)\,\mu_t\,dt\right)}_{\text{P[Survival]}} \underbrace{(Y^i(b_i)\,\mu_{b_i}\,dt)^{\Delta N_i(b_i)}}_{\text{P[Death]}}. \end{aligned}$$

MACRO: Universal Poisson-type likelihood

Poisson-type Models I: Survival Models

M lives, lifetimes $T_1, T_2, \ldots, T_M$, life i observed on $(a_i, b_i]$.

μ_x^θ = Parametric hazard rate on $[0, \infty)$.

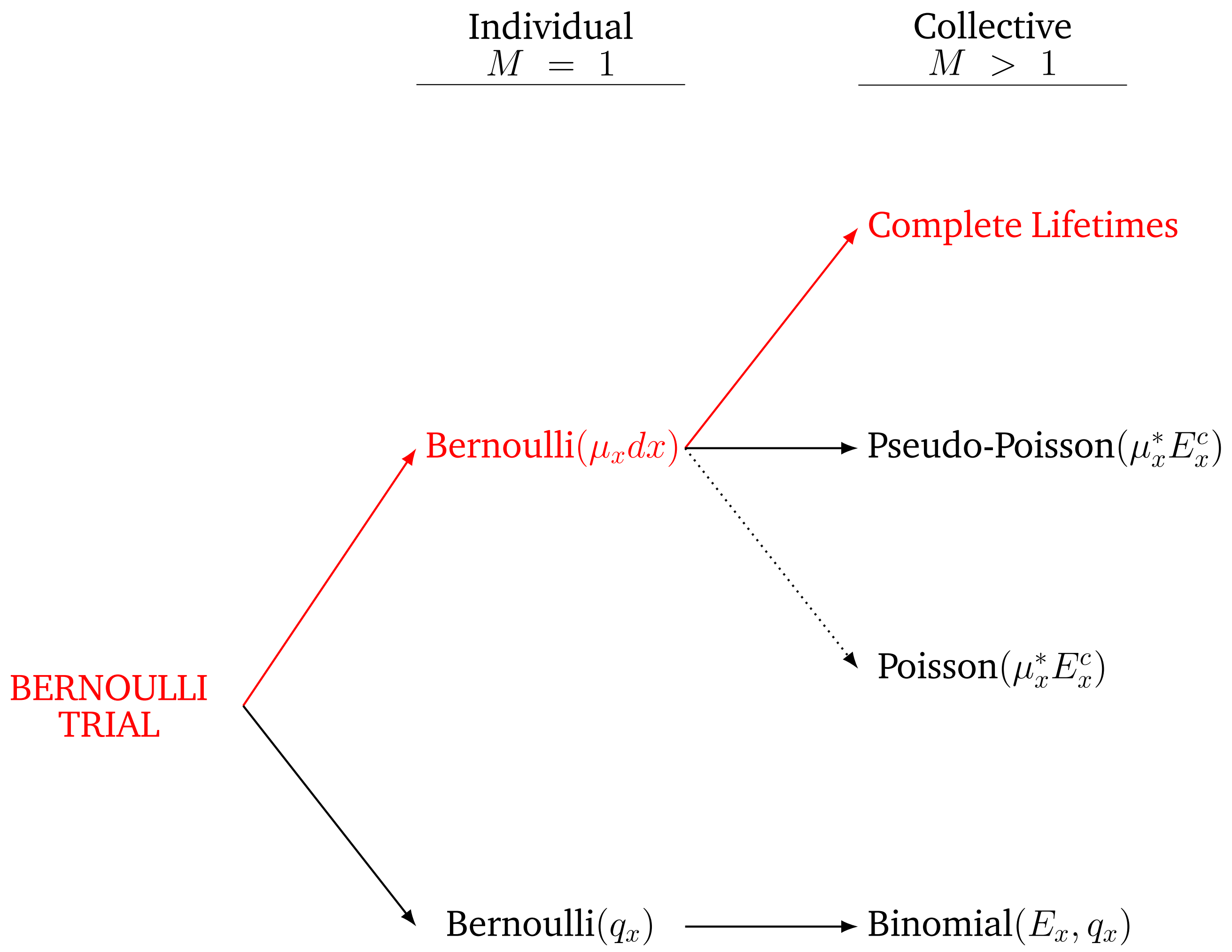
$Y^i(t) = I_{\{\text{Life } i \text{ alive } \textbf{and under observation}\}}$.

$$L = \prod_i L_i \quad = \quad \prod_i \prod_{[0,\infty)} (1 - Y^i(t)\,\mu_t^\theta\,dt)^{(1-\Delta N_i(t))} (Y^i(t)\,\mu_t^\theta\,dt)^{\Delta N_i(t)}$$

$$= \quad \prod_i \exp\left(-\int_{a_i}^{b_i} Y^i(t)\,\mu_t^\theta\,dt\right) (Y^i(b_i)\,\mu_{b_i}^\theta\,dt)^{\Delta N_i(b_i)}.$$

INDIVIDUAL DATA/COMPLETE OBSERVED LIFETIMES/SURVIVAL MODEL

Bernoulli Family Tree (Partial)



Poisson-type Models II: Pseudo-Poisson Models

M lives under observation, life i on $(a_i, b_i]$.

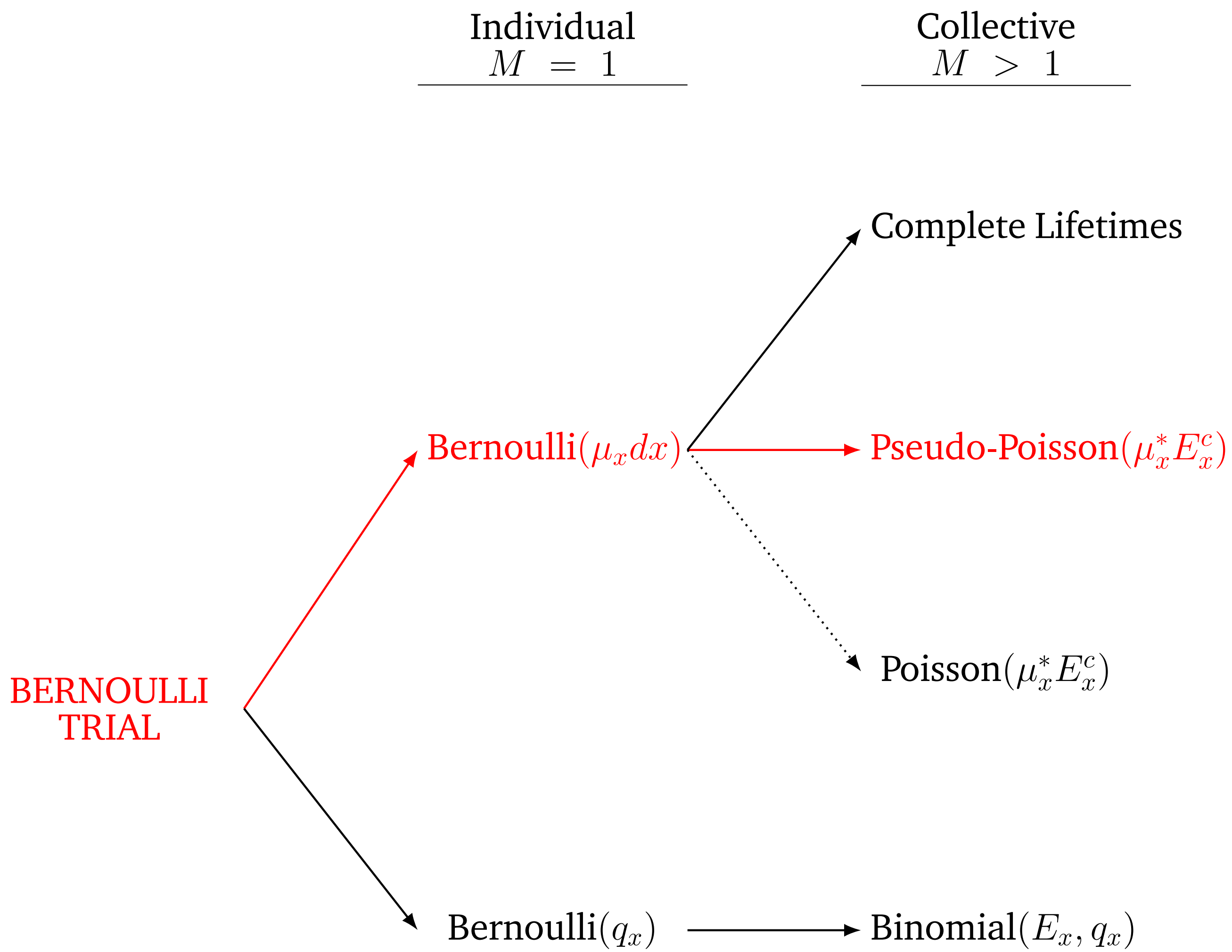
μ_x^* = Constant hazard rate on $(x, x+1]$.

$Y_x^i(t) = I_{\{\text{Life } i \text{ alive } \textbf{and recorded as `active' on } (a_i, b_i] \cap (x, x+1]\}}$.

$$
\begin{aligned}
L = \prod_x \prod_i L_{x,i}^* &= \prod_x \prod_i \prod_{[0,\infty)} (1 - Y_x^i(t)\, \mu_x^*\, dt)^{(1-\Delta N_i(t))} (Y_x^i(t)\, \mu_x^*\, dt)^{\Delta N_i(t)} \\
&= \prod_x \prod_i \exp\left(- \int_x^{x+1} Y_x^i(t)\, \mu_x^*\, dt \right) \prod_{[0,\infty)} (Y_x^i(t)\, \mu_x^*\, dt)^{\Delta N_i(t)} \\
&= \prod_x \exp\left(E_x^c\right) \left(\mu_x^*\right)^{D_x}.
\end{aligned}
$$

M known, E_x^c random variable $\Rightarrow$ GROUPED DATA/PSEUDO-POISSON

Bernoulli Family Tree (Partial)



Poisson-type Models III: True Poisson Models

Random M lives under observation, life i on $(a_i, b_i]$.

μ_x^* = Constant hazard rate on $(x, x+1]$.

$\tilde{Y}_x^i(t) = I_{\{\text{Life } i \text{ alive } \textbf{and recorded as `active' on } (a_i,b_i] \cap (x,x+1]\}}$ constrained so that E_x^c is a pre-determined constant.

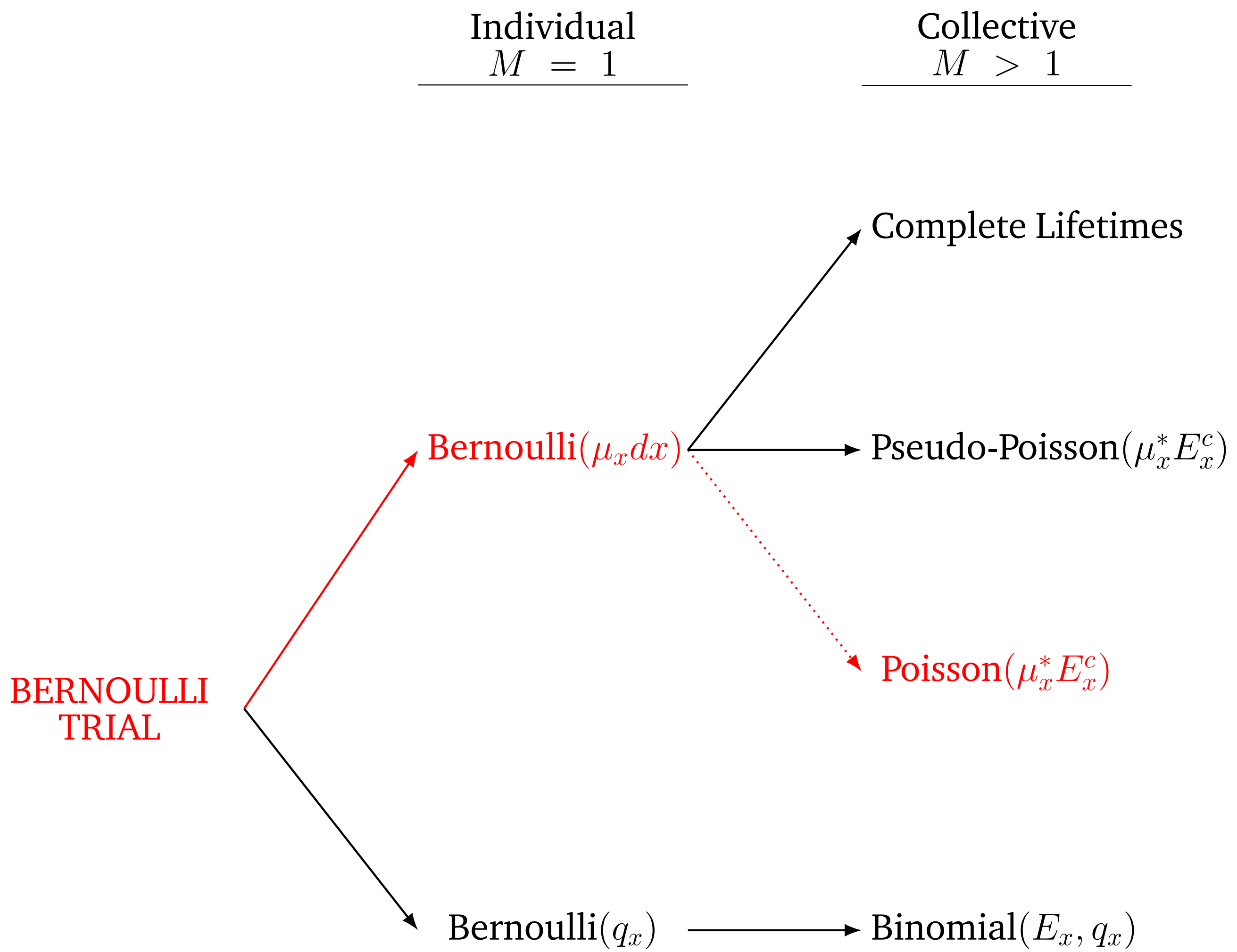
$$L = \prod_x \prod_i L_{x,i}^* \quad \propto \quad \prod_x \prod_i \prod_{[0,\infty)} (1 - \tilde{Y}_x^i(t)\, \mu_x^*\, dt)^{(1-\Delta N_i(t))} (\tilde{Y}_x^i(t)\, \mu_x^*\, dt)^{\Delta N_i(t)}$$

$$= \prod_x \prod_i \exp\left(- \int_x^{x+1} \tilde{Y}_x^i(t)\, \mu_x^*\, dt \right) \prod_{[0,\infty)} (\tilde{Y}_x^i(t)\, \mu_x^*\, dt)^{\Delta N_i(t)}$$

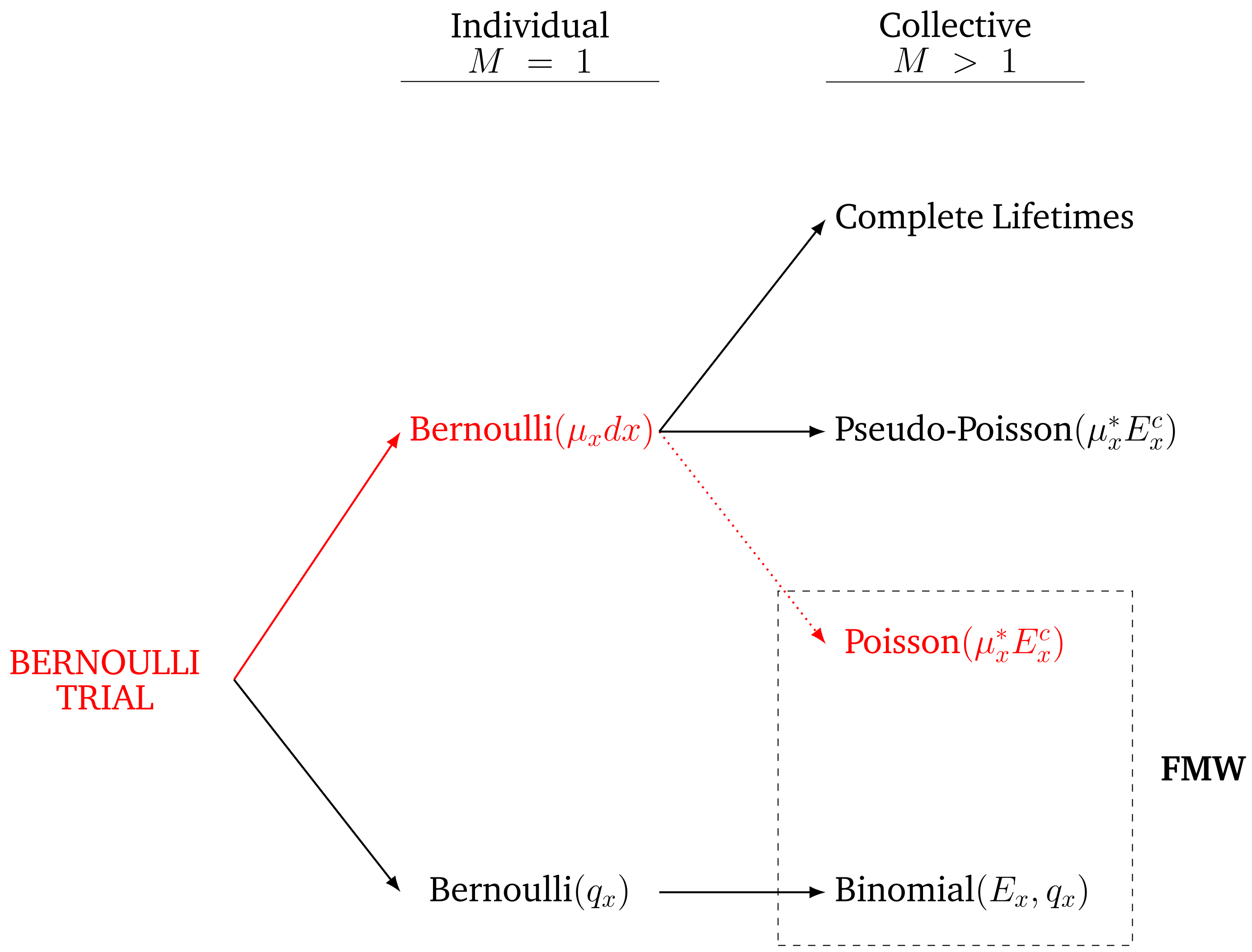
$$= \prod_x \exp\left(E_x^c\right) \left(\mu_x^*\right)^{D_x}.$$

M random variable, E_x^c known $\Rightarrow$ GROUPED DATA/TRUE POISSON

Bernoulli Family Tree (Partial)



Bernoulli Family Tree (Partial)



Continuous-time Models: The Answer

The FAMILY of models with LIKELIHOODS built up from BERNOULLI 'ATOMS' by PRODUCT-INTEGRATION.

- survival models
- pseudo-Poisson models (as per FMW)
- true Poisson models
- multiple-decrement models
- multiple-state models (Markov or not)
- covariates, Cox models etc.
- non-parametric models, Kaplan-Meier etc.
- and so on . . .

Andersen, P.K., Borgan, Ø., Gill, R.D. & Keiding, N. (1993). *Statistical Models Based on Counting Processes*, Springer.